

THE MINIMAL RESULTANT LOCUS

ROBERT RUMELY

ABSTRACT. Let K be a complete, algebraically closed, nonarchimedean valued field, and let $\varphi(z) \in K(z)$ have degree $d \geq 2$. We give an algorithm to determine whether φ has potential good reduction over K , based on a geometric reformulation of the problem using the Berkovich Projective Line. We show the minimal resultant is either achieved at a single point in $\mathbb{P}_{\text{Berk}}^1$, or on a segment, and that minimal resultant locus is contained in the tree in $\mathbb{P}_{\text{Berk}}^1$ spanned by the fixed points and poles of φ . When φ is defined over \mathbb{Q} the algorithm runs in probabilistic polynomial time. If φ has potential good reduction, and is defined over a subfield $H \subset K$, we show there is an $L \subset K$ with $[L : H] \leq (d+1)^2$ such that φ has good reduction over L .

Let K be a complete, algebraically closed nonarchimedean valued field with absolute value $|\cdot|$ and associated valuation $\text{ord}(\cdot) = -\log(|\cdot|)$. Write \mathcal{O} for the ring of integers of K , \mathfrak{M} for its maximal ideal, and \tilde{k} for its residue field.

Let $\varphi(z) \in K(z)$ be a rational function with $\deg(\varphi) = d \geq 1$. Then there are homogeneous polynomials $F(X, Y), G(X, Y) \in K[X, Y]$ of degree d , having no common factor, such that the map $[X : Y] \mapsto [F(X, Y) : G(X, Y)]$ gives the action of φ on \mathbb{P}^1 . After scaling F and G appropriately, one can arrange that F and G belong to $\mathcal{O}[X, Y]$ and that at least one of their coefficients is a unit in \mathcal{O} . Such a pair (F, G) is called a *normalized representation* of φ ; it is unique up to scaling by a unit in \mathcal{O} . Writing $F(X, Y) = f_d X^d + f_{d-1} X^{d-1} Y + \cdots + f_0 Y^d$ and $G(X, Y) = g_d X^d + g_{d-1} X^{d-1} Y + \cdots + g_0 Y^d$, the resultant of F and G is

$$(1) \quad \text{Res}(F, G) = \det \left(\begin{bmatrix} f_d & f_{d-1} & \cdots & f_0 & & & \\ & f_d & f_{d-1} & \cdots & f_0 & & \\ & & & & \vdots & & \\ & & & f_d & f_{d-1} & \cdots & f_0 \\ g_d & g_{d-1} & \cdots & g_0 & & & \\ & g_d & g_{d-1} & \cdots & g_0 & & \\ & & & & \vdots & & \\ & & & g_d & g_{d-1} & \cdots & g_0 \end{bmatrix} \right),$$

and the quantity

$$(2) \quad \text{ordRes}(\varphi) := \text{ord}(\text{Res}(F, G))$$

is independent of the choice of normalized representation. By construction, it is non-negative.

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The *reduction* $\tilde{\varphi}$ is the map $[\tilde{X} : \tilde{Y}] \mapsto [\tilde{F}(\tilde{X}, \tilde{Y}) : \tilde{G}(\tilde{X}, \tilde{Y})]$ on $\mathbb{P}^1(\tilde{k})$ obtained by reducing F and $G \pmod{\mathfrak{M}}$ and eliminating common factors. If $\tilde{\varphi}$ has degree d , then φ is said to have *good reduction*. Likewise, φ is said to have *potential good reduction* if after a change of coordinates by some $\gamma \in \mathrm{GL}_2(K)$, the map $\varphi^\gamma = \gamma^{-1} \circ \varphi \circ \gamma$ has good reduction. It is well known (see e.g. [25], Theorem 2.15) that φ has good reduction if and only if $\mathrm{ordRes}(\varphi) = 0$.

It has been a long-standing problem to find an algorithm to decide whether or not a given φ has potential good reduction. When φ is defined over a local field H_v , Bruin and Molnar ([7]) recently gave an algorithm that determines when φ has potential good reduction over H_v . Their algorithm involves a recursive search, and depends on the fact that H_v is discretely valued.

In this paper we solve the problem by reformulating it in terms of the Berkovich projective line $\mathbb{P}_{\mathrm{Berk}}^1 = \mathbb{P}_{\mathrm{Berk}}^1/K$. We show that the map $\gamma \rightarrow \mathrm{ordRes}(\varphi^\gamma)$ factors through a function $\mathrm{ordRes}_\varphi(\cdot)$ on $\mathbb{P}_{\mathrm{Berk}}^1$ which is continuous, piecewise affine, and convex upwards on each path. It takes on a minimum value. We study the properties of $\mathrm{ordRes}_\varphi(\cdot)$ and the set $\mathrm{MinResLoc}(\varphi) \subset \mathbb{P}_{\mathrm{Berk}}^1$, the *Minimal Resultant Locus*, where its minimal value is attained. We use this to give an algorithm that decides whether φ has potential good reduction and finds a γ for which φ^γ has a minimal resultant. When φ is defined over a subfield $H \subset K$, we obtain an à priori bound of $(d+1)^2$ for the degree of an extension L/H such that there is a $\gamma \in \mathrm{GL}_2(L)$ for which $\mathrm{ordRes}(\varphi^\gamma)$ is minimal.

Recall that $\mathbb{P}_{\mathrm{Berk}}^1$ is a path-connected Hausdorff space containing $\mathbb{P}^1(K)$. By Berkovich's classification theorem (see for example [2], p.5), $\mathbb{P}_{\mathrm{Berk}}^1$ can be viewed as a space whose points correspond to discs in K . There are four types of points: type I points are the points of $\mathbb{P}^1(K)$, which we regard as discs of radius 0. Type II and III points correspond to discs $D(a, r) = \{z \in K : |z - a| \leq r\}$, with type II points corresponding to discs $D(a, r)$ with r in the value group $|K^\times|$, and type III points corresponding to those with $r \notin |K^\times|$. The point ζ_G corresponding to $D(0, 1)$ is called the *Gauss point*. Type IV points serve to complete $\mathbb{P}_{\mathrm{Berk}}^1$; they correspond to (cofinal equivalence classes of) sequences of nested discs with empty intersection. Paths in $\mathbb{P}_{\mathrm{Berk}}^1$ correspond to ascending or descending chains of discs, or unions of chains sharing an endpoint. For example the path from 0 to 1 in $\mathbb{P}_{\mathrm{Berk}}^1$ corresponds to the chains $\{D(0, r) : 0 \leq r \leq 1\}$ and $\{D(1, r) : 1 \geq r \geq 0\}$; here $D(0, 1) = D(1, 1)$. Topologically, $\mathbb{P}_{\mathrm{Berk}}^1$ is a tree: there is a unique path $[x, y]$ between any two points $x, y \in \mathbb{P}_{\mathrm{Berk}}^1$.

The set $\mathbb{H}_{\mathrm{Berk}} = \mathbb{P}_{\mathrm{Berk}}^1 \setminus \mathbb{P}^1(K)$ is called the *Berkovich upper halfspace*; it carries a metric $\rho(x, y)$ called the *logarithmic path distance*, for which the length of the path corresponding to $\{D(a, r) : R_1 \leq r \leq R_2\}$ is $\log(R_2/R_1)$. There are two natural topologies on $\mathbb{P}_{\mathrm{Berk}}^1$, called the *weak* and *strong* topologies. The weak topology on $\mathbb{P}_{\mathrm{Berk}}^1$ is the coarsest one which makes the evaluation functionals $z \rightarrow |f(z)|$ continuous for all $f(z) \in K(z)$; under the weak topology, $\mathbb{P}_{\mathrm{Berk}}^1$ is compact and $\mathbb{P}^1(K)$ is dense in it. The basic open sets for the weak topology are the path-components of $\mathbb{P}_{\mathrm{Berk}}^1 \setminus \{P_1, \dots, P_n\}$ as $\{P_1, \dots, P_n\}$ ranges over finite subsets of $\mathbb{H}_{\mathrm{Berk}}$. The strong topology on $\mathbb{P}_{\mathrm{Berk}}^1$ (which is finer than the weak topology) restricts to the topology on $\mathbb{H}_{\mathrm{Berk}}$ induced by $\rho(x, y)$. The basic open sets for the strong topology are the $\rho(x, y)$ -balls in $\mathbb{H}_{\mathrm{Berk}}$, together with the basic open sets from the weak topology. Type II points are dense in $\mathbb{P}_{\mathrm{Berk}}^1$ for both topologies. The action of φ on $\mathbb{P}^1(K)$ extends functorially to an action on $\mathbb{P}_{\mathrm{Berk}}^1$, which is continuous for

both topologies, and takes points of a given type to points of the same type. Similarly, the action of $\mathrm{GL}_2(K)$ on $\mathbb{P}^1(K)$ extends to an action on $\mathbb{P}_{\mathrm{Berk}}^1$, which is continuous for both topologies, and preserves the type of each point. The action of $\mathrm{GL}_2(K)$ also preserves the logarithmic path distance: $\rho(\gamma(x), \gamma(y)) = \rho(x, y)$ for all $x, y \in \mathbb{H}_{\mathrm{Berk}}$ and all $\gamma \in \mathrm{GL}_2(K)$. For these and other facts, see ([2]) and ([3], [4], [11], [12], [14], [23]).

It follows from standard formulas for the resultant (see for example (Silverman [25], Exercise 2.7, p.75)) that for each $\gamma \in \mathrm{GL}_2(K)$ and each $\tau \in K^\times \cdot \mathrm{GL}_2(\mathcal{O})$, we have

$$\mathrm{ordRes}(\varphi^\gamma) = \mathrm{ordRes}(\varphi^{\gamma\tau}) .$$

On the other hand, $\mathrm{GL}_2(K)$ acts transitively on type II points, and $K^\times \cdot \mathrm{GL}_2(\mathcal{O})$ is the stabilizer of the Gauss point. This means there is a well-defined function $\mathrm{ordRes}_\varphi(\cdot)$ on the type II points in $\mathbb{P}_{\mathrm{Berk}}^1$, given by

$$(3) \quad \mathrm{ordRes}_\varphi(\gamma(\zeta_G)) := \mathrm{ordRes}(\varphi^\gamma) .$$

This observation is the key to our investigation. Our main result is

Theorem 0.1 (Main Theorem). *Suppose $d = \deg(\varphi) \geq 2$. The function $\mathrm{ordRes}_\varphi(\cdot)$ on type II points extends uniquely to a function $\mathrm{ordRes}_\varphi : \mathbb{P}_{\mathrm{Berk}}^1 \rightarrow [0, \infty]$ continuous with respect to the strong topology. On each path in $\mathbb{P}_{\mathrm{Berk}}^1$, it is piecewise affine and convex upwards with respect to the logarithmic path distance. It is finite on $\mathbb{H}_{\mathrm{Berk}}$ and ∞ on $\mathbb{P}^1(K)$. It achieves a minimum on $\mathbb{P}_{\mathrm{Berk}}^1$. The set $\mathrm{MinResLoc}(\varphi)$ where $\mathrm{ordRes}_\varphi(\cdot)$ takes on its minimum is contained in the tree $\Gamma_{\mathrm{Fix}, \varphi^{-1}(\infty)}$ spanned by the fixed points and poles of φ in $\mathbb{P}^1(K)$, and lies in $\{z \in \mathbb{H}_{\mathrm{Berk}} : \rho(\zeta_G, z) \leq \frac{2}{d-1} \mathrm{ordRes}(\varphi)\}$. $\mathrm{MinResLoc}(\varphi)$ consists of a single type II point if d is even, and is a type II point or a segment with type II endpoints if d is odd. If the minimum value of $\mathrm{ordRes}_\varphi(\cdot)$ is 0 (that is, if φ has potential good reduction), then $\mathrm{MinResLoc}(\varphi)$ consists of a single point.*

In the proof of Theorem 0.1, one sees that each affine piece of $\mathrm{ordRes}_\varphi(\cdot)$ has an integer slope $m \equiv d^2 + d \pmod{2d}$ with $-d^2 - d \leq m \leq d^2 + d$, and that breaks between affine pieces occur at type II points. By Proposition 3.5, in Theorem 0.1 the tree $\Gamma_{\mathrm{Fix}, \varphi^{-1}(\infty)}$ can be replaced by the tree $\Gamma_{\mathrm{Fix}, \varphi^{-1}(a)}$ spanned by the fixed points and the preimages of a , for any $a \in \mathbb{P}^1(K)$. The Theorem has the following consequences:

(1) Relative to computations in K , there is an algorithm (Algorithm A) to determine whether or not φ has potential good reduction. If it does, one can find a $\gamma \in \mathrm{GL}_2(K)$ such that φ^γ has good reduction.

Indeed, the algorithm is as follows. First, find the fixed points $\{P_0, \dots, P_d\}$ and poles $\{Q_1, \dots, Q_d\}$ of φ . Choose one of the fixed points, say P_0 , and restrict $\mathrm{ordRes}_\varphi(\cdot)$ in turn to each of the $2d$ paths $[P_0, P_k]$ and $[P_0, Q_k]$ for $k = 1, \dots, d$. The resulting piecewise affine functions can be computed and their minima found. If the minimum value on some path is 0, then φ has good reduction at the corresponding point. If all minima are positive, then φ does not have potential good reduction. When φ is defined over \mathbb{Q} , Algorithm A can be implemented to run in probabilistic polynomial time.

When φ is defined over a local field H_v , we give another algorithm (Algorithm B) which minimizes $\mathrm{ordRes}(\varphi^\gamma)$ for $\gamma \in \mathrm{GL}_2(H_v)$. This algorithm is based on steepest descent, and runs in probabilistic polynomial time. It answers the same question as the Bruin-Molnar algorithm, but is more conceptual, and should be more efficient. However, the two algorithms have many aspects in common.

(2) If φ is defined over a subfield $H \subset K$, there is an à priori bound of $(d+1)^2$ for the degree of an extension L/H such that $\text{ordRes}(\varphi^\gamma)$ is minimal for some $\gamma \in \text{GL}_2(L)$ (see Theorem 3.6). In particular, if φ has potential good reduction, this is a bound for the degree of an extension where it achieves good reduction. It follows from this that if H is Henselian (in particular, if H is complete), the statement “ φ has potential good reduction” is first-order in the theory of H , in the sense of mathematical logic.

(3) The Minimal Resultant Locus can be a segment of positive length (see Examples 2.5 and 2.7). Hence there can be fundamentally different coordinate changes (that is, coordinate changes by γ 's belonging to different cosets of $K^\times \cdot \text{GL}_2(\mathcal{O})$) for which φ^γ has minimal resultant. However, this can only happen when d is odd and φ does not have potential good reduction.

(4) If φ is defined over a subfield $H \subset K$, and φ has potential good reduction, let H_φ be the intersection of all fields L with $H \subset L \subset K$ such that φ^γ has good reduction for some $\gamma \in \text{GL}_2(L)$ (the ‘field of moduli for the good reduction problem’). We give examples where $H_\varphi = H$ but φ^γ does not have good reduction for any $\gamma \in \text{GL}_2(H)$. Thus there need not be a unique minimal extension L/H where φ achieves good reduction.

(5) Suppose H is a number field. An elliptic curve E/H has a global minimal model over H if and only if a certain class $[\mathbf{a}_E]$ in the ideal class group of \mathcal{O}_H , the *Weierstrass class*, is principal. When $\varphi(z) \in H(z)$ and $\deg(\varphi) \geq 2$, Silverman has constructed an ideal class $[\mathbf{a}_\varphi]$ such that if φ has global minimal model over H , then $[\mathbf{a}_\varphi]$ is trivial (see [25], Proposition 4.99). He asks if the converse is true ([25], p.237, Exercise 4.4.6(c)). We give examples of number fields H and functions $\varphi(z) \in H(z)$ for which $[\mathbf{a}_\varphi]$ is trivial but φ has no global minimal model.

Our second result concerns the stability of $\text{ordRes}_\varphi(\cdot)$ and $\text{MinResLoc}(\varphi)$ under perturbations of φ . It also specifies the precision needed for numerical implementations of Algorithms A and B.

Theorem 0.2. *Suppose $\varphi(z), \tilde{\varphi}(z) \in K(z)$ have degree $d \geq 2$, with normalized representations $(F, G), (\tilde{F}, \tilde{G})$ respectively. Put $R = \text{ordRes}(\varphi)$, and let $M > 0$ be arbitrary. If*

$$(4) \quad \min(\text{ord}(\tilde{F} - F), \text{ord}(\tilde{G} - G)) > \max\left(R, \frac{1}{2d}(R + (d^2 + d)M)\right),$$

then $\text{ordRes}_\varphi(\xi) = \text{ordRes}_{\tilde{\varphi}}(\xi)$ for all ξ with $\rho(\zeta_G, \xi) \leq M$. Let $f(d) = \frac{2d^2 + 3d - 1}{2d^2 - 2d}$. If

$$(5) \quad \min(\text{ord}(\tilde{F} - F), \text{ord}(\tilde{G} - G)) > f(d) \cdot R,$$

then $\text{MinResLoc}(\varphi) = \text{MinResLoc}(\tilde{\varphi})$, and $\text{ordRes}_\varphi(\xi) = \text{ordRes}_{\tilde{\varphi}}(\xi)$ for all ξ with $\rho(\zeta_G, \xi) \leq \frac{2}{d-1}\text{ordRes}(\varphi)$.

Note that $f(2) = 3.25$, $f(3) = 2.166\ldots$, and $1 < f(d) < 2$ for $d \geq 4$.

The structure of the paper is as follows. In Section 1 we prove Theorems 0.1 and 0.2. In Section 2 we give examples illustrating various phenomena which occur. In Section 3 we give applications of the theory. In Section 4 we present Algorithms A and B. Finally, in Section 5 we prove an analogue of Theorem 0.1 when $d = 1$.

1. PROOF OF THE MAIN THEOREMS

In this section we establish Theorems 0.1 and 0.2. Suppose $\varphi(z) \in K(z)$ has degree d . Then

$$\varphi(z) = \frac{F(z, 1)}{G(z, 1)}$$

where $F(X, Y) = f_d X^d + f_{d-1} X^{d-1} Y + \cdots + f_0 Y^d$ and $G(X, Y) = g_d X^d + g_{d-1} X^{d-1} Y + \cdots + g_0 Y^d$ are homogeneous polynomials in $K[X, Y]$ of degree d with no common factor. The pair (F, G) is called a *representation* of φ ; it is unique up to scaling by a nonzero constant. Put $\text{ord}(F) = \min_{0 \leq i \leq d}(\text{ord}(f_i))$, $\text{ord}(G) = \min_{0 \leq i \leq d}(\text{ord}(g_i))$.

The resultant of F and G is defined by the $2d \times 2d$ determinant in formula (1). For any $c \in K^\times$, we have $\text{Res}(cF, cG) = c^{2d} \text{Res}(F, G)$. By choosing c so that $\text{ord}(c) = \min(\text{ord}(F), \text{ord}(G))$ and replacing (F, G) by $(c^{-1}F, c^{-1}G)$ we can assume that

$$\min(\text{ord}(F), \text{ord}(G)) = 0;$$

in this case (F, G) is called a *normalized representation* of φ , and $\text{ordRes}(\varphi)$ is defined to be $\text{ord}(\text{Res}(F, G))$ as in (2). Clearly $\text{ordRes}(\varphi)$ is independent of the choice of normalized representation, and $\text{ordRes}(\varphi) \geq 0$.

Whether or not (F, G) is normalized, we have

$$(6) \quad \text{ordRes}(\varphi) = \text{ord}(\text{Res}(F, G)) - 2d \min(\text{ord}(F), \text{ord}(G)).$$

Given $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{GL}_2(K)$, let $\text{Adj}(\gamma) = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$ and define (F^γ, G^γ) by

$$(7) \quad \begin{bmatrix} F^\gamma(X, Y) \\ G^\gamma(X, Y) \end{bmatrix} = \text{Adj}(\gamma) \circ \begin{bmatrix} F \\ G \end{bmatrix} \circ \gamma \circ \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} DF(AX + BY) - BG(CX + DY) \\ -CF(AX + BY) + AG(CX + DY) \end{bmatrix}.$$

Then (F^γ, G^γ) is a homogeneous representation of φ^γ . It is known (see ([25], Exercise 2.7(c), p.76) that $\text{Res}(F^\gamma, G^\gamma) = \text{Res}(F, G) \cdot \det(\gamma)^{d^2+d}$, so

$$(8) \quad \text{ordRes}(\varphi^\gamma) = \text{ordRes}(F, G) + (d^2 + d) \text{ord}(\det(\gamma)) - 2d \min(\text{ord}(F^\gamma), \text{ord}(G^\gamma)).$$

We will prove Theorems 0.1 and 0.2 after a series of preliminary results. In Theorem 0.1 it is assumed that $d \geq 2$; however, for use in §5, we will develop the theory for $d \geq 1$, and make explicit the places where $d \geq 2$ is used.

We begin by recalling some facts about the action of $\text{GL}_2(K)$ on $\mathbb{P}_{\text{Berk}}^1$.

Proposition 1.1. *The natural action of $\text{GL}_2(K)$ on $\mathbb{P}^1(K)$ extends to an action on $\mathbb{P}_{\text{Berk}}^1$ such that*

- (A) *The stabilizer of ζ_G in $\text{GL}_2(K)$ is $K^\times \cdot \text{GL}_2(\mathcal{O})$;*
- (B) *For each $\gamma \in \text{GL}_2(K)$, one has $\rho(\gamma(x), \gamma(y)) = \rho(x, y)$ for all $x, y \in \mathbb{H}_{\text{Berk}}$;*
- (C) *For each $\gamma \in \text{GL}_2(K)$ and each path $[x, y]$, one has $\gamma([x, y]) = [\gamma(x), \gamma(y)]$;*
- (D) *For any triple (a_0, A, a_1) where $a_0, a_1 \in \mathbb{P}(K)$, $a_0 \neq a_1$, and A is a type II point in $[x, y]$, if (b_0, B, b_1) is another triple of the same kind, there is a $\gamma \in \text{GL}_2(K)$ such that $\gamma(a_0) = b_0$, $\gamma(A) = B$, and $\gamma(a_1) = b_1$. In particular, $\text{GL}_2(K)$ acts transitively on the type II points in $\mathbb{P}_{\text{Berk}}^1$.*

Proof. As discussed in ([2], §2.3), the natural action of any rational function $f(z) \in K(z)$ on $\mathbb{P}^1(K)$ extends uniquely to a continuous action on $\mathbb{P}_{\text{Berk}}^1$. For part (A), suppose $\gamma \in \text{GL}_2(K)$ stabilizes ζ_G , and let $\gamma(0) = a$, $\gamma(1) = b$, $\gamma(\infty) = c$. By ([2], Lemma 2.17)

$\gamma(z)$ has nonconstant reduction, so the reductions \bar{a} , \bar{b} , and \bar{c} are distinct in $\mathbb{P}^1(\tilde{k})$. If none of $\bar{a}, \bar{b}, \bar{c}$ is ∞ , then

$$(9) \quad \gamma_0(z) = \frac{cz - a(b - c)/(b - a)}{z - (b - c)/(b - a)}$$

belongs to $\mathrm{GL}_2(\mathcal{O})$ and satisfies $\gamma_0(0) = a$, $\gamma_0(1) = b$, $\gamma_0(\infty) = c$. If one of the reductions is ∞ , by making simple modifications to (9) one still finds a $\gamma_0 \in \mathrm{GL}_2(\mathcal{O})$ with $\gamma_0(0) = a$, $\gamma_0(1) = b$, $\gamma_0(\infty) = c$. Since $\gamma_0^{-1} \circ \gamma \in \mathrm{GL}_2(K)$ fixes three points in $\mathbb{P}^1(K)$, it must be a multiple of the identity matrix. Part (B) is ([2], Proposition 2.30). Part (C) follows from the fact that if $\gamma \in \mathrm{GL}_2(K)$, the action of γ on $\mathbb{P}_{\mathrm{Berk}}^1$ must be bijective and bicontinuous, since $\gamma^{-1} \circ \gamma = \gamma \circ \gamma^{-1} = \mathrm{id}$. Part (D) is ([2], Corollary 2.13 (B)). \square

Lemma 1.2. *For any distinct points $x, y \in \mathbb{P}^1(K)$, the function $\mathrm{ordRes}_\varphi(\cdot)$ on type II points extends to a continuous function on the path $[x, y]$, which is piecewise affine with respect to the logarithmic path distance, and convex up. The extension is finite on $[x, y] \cap \mathbb{H}_{\mathrm{Berk}}$, and when $d \geq 2$, it is ∞ at x and y .*

If H is a field of definition for φ (so $H(x, y)$ is a field of definition for φ , x , and y), then each affine piece of $\mathrm{ordRes}_\varphi(\cdot)$ has the form $mt + c$ for some integer m in the range $-d^2 - d \leq m \leq d^2 + d$ satisfying $m \equiv d^2 + d \pmod{2d}$, and some number c in the value group $\mathrm{ord}(H(x, y)^\times)$, where t is a parameter measuring the logarithmic path distance along $[x, y]$. There are at most $d + 1$ distinct affine pieces, and the breaks between affine pieces occur at type II points.

Proof. Fix $\gamma \in \mathrm{GL}_2(K)$ with $\gamma(0) = x$ and $\gamma(\infty) = y$. The action of $\mathrm{GL}_2(K)$ on $\mathbb{P}_{\mathrm{Berk}}^1$ takes paths to paths, so $\gamma([0, \infty]) = [x, y]$. The type II points on $[0, \infty]$ are the points $\zeta_{|A|}$ corresponding to discs $D(0, |A|)$, as A runs over elements of K^\times , and if we put $\mu_A = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(K)$, then $\zeta_{|A|} = \mu_A(\zeta_G)$. Now let $\gamma_A = \gamma \circ \mu_A$. As A varies, the type II points on $[x, y]$ are the points $\gamma(\zeta_{|A|}) = \gamma_A(\zeta_G)$, and for all $A, B \in K^\times$ we have

$$\rho(\gamma(\zeta_{|A|}), \gamma(\zeta_{|B|})) = |\mathrm{ord}(A) - \mathrm{ord}(B)|.$$

Write

$$(10) \quad \begin{aligned} F^\gamma(X, Y) &= a_d X^d + a_{d-1} X^{d-1} Y + \cdots + a_0 Y^d, \\ G^\gamma(X, Y) &= b_d X^d + b_{d-1} X^{d-1} Y + \cdots + b_0 Y^d. \end{aligned}$$

Since $\varphi^{\gamma_A} = (\varphi^\gamma)^{\mu_A}$ we have $\begin{bmatrix} F^{\gamma_A}(X, Y) \\ G^{\gamma_A}(X, Y) \end{bmatrix} = \begin{bmatrix} F^\gamma(AX, Y) \\ A G^\gamma(AX, Y) \end{bmatrix}$; thus

$$(11) \quad \begin{aligned} F^{\gamma_A}(X, Y) &= A^d a_d X^d + A^{d-1} a_{d-1} X^{d-1} Y + \cdots + a_0 Y^d, \\ G^{\gamma_A}(X, Y) &= A^{d+1} b_d X^d + A^d b_{d-1} X^{d-1} Y + \cdots + A b_0 Y^d. \end{aligned}$$

Put $Q_A = \gamma_A(\zeta_G)$ and write $t = \text{ord}(A)$. Since $\det(\gamma_A) = A \det(\gamma)$, it follows from formula (8) that

$$\begin{aligned}
 \text{ordRes}_\varphi(Q_A) &= \text{ordRes}(\varphi^{\gamma_A}) \\
 (12) \quad &= \text{ordRes}(F^\gamma, G^\gamma) + (d^2 + d)\text{ord}(A) \\
 &\quad - 2d \min(\text{ord}(a_0), \dots, \text{ord}(A^d a_d), \text{ord}(A b_0), \dots, \text{ord}(A^{d+1} b_d)) \\
 (13) \quad &= \max \left(\max_{0 \leq \ell \leq d} ((d^2 + d - 2d\ell)t + C_\ell), \max_{0 \leq \ell \leq d} ((d^2 + d - 2d(\ell + 1))t + D_\ell) \right),
 \end{aligned}$$

where $C_\ell = \text{ordRes}(F^\gamma, G^\gamma) - 2d \text{ord}(a_\ell)$, $D_\ell = \text{ordRes}(F^\gamma, G^\gamma) - 2d \text{ord}(b_\ell)$.

Now let t vary over \mathbb{R} . Since the type II points Q_A (which correspond to values of t in the divisible group $\text{ord}(K^\times)$) are dense in $[x, y]$ for the path distance topology, we can use the right side of (13) to extend $\text{ordRes}_\varphi(\cdot)$ continuously to $[x, y]$, omitting any terms in (13) for which C_ℓ or D_ℓ is $-\infty$ (such terms correspond to coefficients a_ℓ or b_ℓ which are 0). Clearly the extension, being the maximum of finitely many affine functions of t , is piecewise affine and convex upwards. Now suppose $d \geq 2$. Since $F(X, Y)$ and $G(X, Y)$ have no common factors, the same is true for $F^\gamma(X, Y)$ and $G^\gamma(X, Y)$; it follows that at least one of a_0, b_0 is nonzero, and at least one of a_d, b_d is nonzero. The slopes of the corresponding affine functions are $d^2 + d$, $d^2 - d$, $-(d^2 - d)$ and $-(d^2 + d)$; since $d \geq 2$ these are all nonzero. Thus at least one of the affine functions in (13) has positive slope and at least one has negative slope; this means the extended function $\text{ordRes}_\varphi(\cdot)$ is finite on $[x, y] \cap \mathbb{H}_{\text{Berk}}$, and is ∞ at x and y .

Let H be a field of definition for φ . Then $F(X, Y)$, $G(X, Y)$ can be taken to be rational over H , and γ can be taken to be rational over $H(x, y)$; if this is the case then $a_0, \dots, a_d, b_0, \dots, b_d$ and $\det(\gamma)$ will also be rational over $H(x, y)$. Comparing (12) and (13) we see that each affine piece of $\text{ordRes}_\varphi(\cdot)$ has the form $mt + c$, where m is an integer in the range $-d^2 - d \leq m \leq d^2 + d$ satisfying $m \equiv d^2 + d \pmod{2d}$, and c belongs to the value group $\text{ord}(H(x, y)^\times)$. If two of the affine functions in (13) have the same slope, only one will contribute to $\text{ordRes}_\varphi(\cdot)$. There are $d + 1$ possible slopes, so $\text{ordRes}_\varphi(\cdot)$ has at most $d + 1$ affine pieces on $[x, y]$.

Finally, suppose $m_i t + c_i$ and $m_j t + c_j$ are consecutive affine pieces. Their intersection occurs at

$$(14) \quad t = t_{ij} = -\frac{c_j - c_i}{m_j - m_i}$$

which belongs to $\text{ord}(K^\times)$; thus the breaks between affine pieces occur at type II points. Indeed, $m = m_j - m_i$ is a nonzero integer satisfying $m \equiv 0 \pmod{2d}$, with $|m| \leq 2d(d+1)$; and that by (12) and (13) $c_j - c_i \in 2d \cdot \text{ord}(H(x, y)^\times)$. Thus t_{ij} actually belongs to the divisible hull of $\text{ord}(H(x, y)^\times)$, with denominator taken from $\{1, 2, \dots, d+1\}$. \square

Proposition 1.3. *There is a unique extension of $\text{ordRes}_\varphi(\cdot)$ on type II points to a function $\text{ordRes}_\varphi : \mathbb{P}_{\text{Berk}}^1 \rightarrow [0, \infty]$ which agrees with the one given in Lemma 1.2 on paths with endpoints in $\mathbb{P}^1(K)$, and is continuous on \mathbb{H}_{Berk} for the strong topology. When $d = 1$, the extension is continuous with respect to the strong topology at each $x \in \mathbb{H}_{\text{Berk}}$, and at each $x \in \mathbb{P}^1(K)$ where $\text{ordRes}_\varphi(x) = \infty$. When $d \geq 2$, it is continuous with respect to the strong topology at each $x \in \mathbb{P}_{\text{Berk}}^1$. The extension is finite on \mathbb{H}_{Berk} , and when $d \geq 2$ it takes the value ∞ at each $x \in \mathbb{P}^1(K)$.*

On each path in $\mathbb{P}_{\text{Berk}}^1$, the extension is convex upwards and piecewise affine with respect to $\rho(x, y)$; moreover, the slope of each affine piece is an integer $m \equiv d^2 + d \pmod{2d}$ with $-d^2 - d \leq m \leq d^2 + d$, the breaks between affine pieces occur at type II points, and there are at most $d + 1$ distinct affine pieces. In particular, on \mathbb{H}_{Berk} , the extension is Lipschitz continuous with respect to $\rho(x, y)$ with Lipschitz constant $d^2 + d$.

Proof. Given two paths $[x_1, y_1]$, $[x_2, x_2]$ with endpoints in $\mathbb{P}^1(K)$, the extensions of $\text{ordRes}_\varphi(\cdot)$ to $[x_1, y_1]$ and $[x_2, x_2]$ given by Lemma 1.2 are consistent on $[x_1, y_1] \cap [x_2, x_2]$, since type II points are dense in the intersection if it is nonempty, and the extension to each path is continuous. Define $\text{ordRes}_\varphi(\cdot)$ to be the extension given by Lemma 1.2 on each path $[x, y]$ with endpoints in $\mathbb{P}^1(K)$. In this way, we obtain a well-defined function $\text{ordRes}_\varphi(\cdot)$ on the points of type I, II, and III in $\mathbb{P}_{\text{Berk}}^1$. When $d \geq 2$, Lemma 1.2 shows that $\text{ordRes}_\varphi(x) = \infty$ for each $x \in \mathbb{P}^1(K)$.

We next show that there is a unique continuous extension of $\text{ordRes}_\varphi(\cdot)$ to type IV points. Since any pair of type II points belongs to a path with endpoints in $\mathbb{P}^1(K)$, Lemma 1.2 shows that for all type II points x, y we have

$$|\text{ordRes}_\varphi(x) - \text{ordRes}_\varphi(y)| \leq (d^2 + d) \cdot \rho(x, y).$$

Since each point of IV is at finite logarithmic path distance from ζ_G , and type II points are dense in \mathbb{H}_{Berk} with respect to $\rho(x, y)$, there is a unique extension of $\text{ordRes}_\varphi(\cdot)$ to \mathbb{H}_{Berk} which is Lipschitz continuous with respect to $\rho(x, y)$, with Lipschitz constant $d^2 + d$. Since $\text{ordRes}_\varphi(x) \geq 0$ on type II points, $\text{ordRes}_\varphi(z) \geq 0$ for all $z \in \mathbb{P}_{\text{Berk}}^1$.

Since each segment $[u, v]$ with type II endpoints is contained in a path $[x, y]$ with type I endpoints, the restriction of $\text{ordRes}_\varphi(\cdot)$ to $[u, v]$ is piecewise affine and convex upwards with respect to the logarithmic path distance, with most $d + 1$ affine pieces, and slopes $m \equiv d^2 + d \pmod{2d}$ where $-d^2 - d \leq m \leq d^2 + d$; the breaks between affine pieces occur at type II points. These same properties must hold for $\text{ordRes}_\varphi(\cdot)$ on an arbitrary path $[z, w]$ in $\mathbb{P}_{\text{Berk}}^1$, since the interior of the path can be exhausted by an increasing sequence of segments with type II endpoints, and the number of affine pieces on each such segment is uniformly bounded.

To complete the proof, it suffices to show that $\text{ordRes}_\varphi(\cdot)$ is continuous with respect to the strong topology at each type I point x where $\text{ordRes}_\varphi(x) = \infty$. Fix $y \in \mathbb{P}^1(K)$ with $y \neq x$, and consider the path $[x, y]$. For each $P \in [x, y] \cap \mathbb{H}_{\text{Berk}}$, let $U_x(P)$ be the component of $\mathbb{P}_{\text{Berk}}^1 \setminus \{P\}$ containing x . As $P \rightarrow x$, the sets $U_x(P)$ form a basis for the neighborhoods of x in the strong topology. We claim that for each $M \in \mathbb{R}$, there is a P_M such that $\text{ordRes}_\varphi(z) > M$ for all $z \in U_x(P_M)$. To see this, note that since $\text{ordRes}_\varphi(P)$ increases to ∞ as $P \rightarrow x$ along $[x, y]$, there is a P_M such that $\text{ordRes}_\varphi(P_M) > M$ and $\text{ordRes}_\varphi(\cdot)$ is increasing on $[P_M, x]$. Let $z \in U_x(P_M)$ be arbitrary. The path $[P_M, z]$ shares an initial segment with $[P_M, x]$, and $\text{ordRes}_\varphi(\cdot)$ is increasing along that initial segment. Since $\text{ordRes}_\varphi(\cdot)$ is convex up on $[P_M, z]$, we have $\text{ordRes}_\varphi(z) > \text{ordRes}_\varphi(P_M) > M$. \square

For each $Q \in \mathbb{P}_{\text{Berk}}^1$, we call paths $[Q, x]$ and $[Q, y]$ emanating from Q *equivalent* if they share an initial segment. The *tangent space* T_Q is the set of equivalence classes of paths emanating from Q ; these classes are called *directions*. The directions at Q are in 1 – 1 correspondence with the components of $\mathbb{P}_{\text{Berk}}^1 \setminus \{Q\}$. If Q is of type I or IV, T_Q has one element; if Q is of type III, T_Q has two elements; and if Q is of type II, T_Q is

infinite. Given $\beta \neq Q$, we will write $\vec{v}_\beta \in T_Q$ for the direction containing $[Q, \beta]$, or $\vec{v}_{Q,\beta}$ if it is necessary to specify Q .

Recall that $\tilde{k} = \mathcal{O}/\mathfrak{M}$ is the residue field of K . When $Q = \zeta_G$, the components of $\mathbb{P}_{\text{Berk}}^1 \setminus \{\zeta_Q\}$ correspond to elements of $\mathbb{P}^1(\tilde{k})$; thus the directions in T_{ζ_G} are \vec{v}_∞ and the \vec{v}_β for $\beta \in \mathcal{O}$, where $\vec{v}_{\beta_1} = \vec{v}_{\beta_2}$ iff $\beta_1 \equiv \beta_2 \pmod{\mathfrak{M}}$. For an arbitrary type II point Q , we can write $Q = \gamma(\zeta_G)$ for some $\gamma \in \text{GL}_2(K)$; since γ takes paths to paths, it induces a 1-1 correspondence $\gamma_* : T_{\zeta_G} \rightarrow T_Q$ with $\gamma_*(\vec{v}_\beta) = \vec{v}_{\gamma(\beta)} \in T_Q$. Hence the directions in T_Q are $\vec{v}_{\gamma(\infty)}$ and the $\vec{v}_{\gamma(\beta)}$ for $\beta \in \mathcal{O}$, where again $\vec{v}_{\gamma(\beta_1)} = \vec{v}_{\gamma(\beta_2)}$ iff $\beta_1 \equiv \beta_2 \pmod{\mathfrak{M}}$.

We will say $\text{ordRes}_\varphi(\cdot)$ is *locally decreasing* (resp. *locally constant*, resp. *increasing*) in a direction \vec{v} at Q if it is initially decreasing (resp. constant, resp. increasing) along $[Q, \beta]$ for some (hence every) path with $\vec{v} = \vec{v}_\beta$. A crucial observation is that since $\text{ordRes}_\varphi(\cdot)$ is convex upward, at each point Q there can be at most one direction in which $\text{ordRes}_\varphi(\cdot)$ is locally decreasing: thus, $\text{ordRes}_\varphi(\cdot)$ satisfies the principle of steepest descent. Likewise, if it is locally constant in some direction at Q , it must be locally constant or increasing in every other direction. If it is locally increasing in some direction at Q , by convexity it must be increasing along every path $[Q, \beta]$ in that direction, so we do not distinguish between *locally increasing* and *increasing*.

When Q is of type II, we will now give necessary and sufficient conditions for $\text{ordRes}_\varphi(\cdot)$ to be locally decreasing, locally constant, or increasing in a given direction. Suppose $Q = \gamma(\zeta_G)$ where $\gamma \in \text{GL}_2(K)$; let (F^γ, G^γ) be the representation of φ^γ from (7). By replacing γ with $c\gamma$ for an appropriate $c \in K^\times$ (which does not change action of γ) we can assume (F^γ, G^γ) is normalized. As in (10), write

$$(15) \quad \begin{aligned} F^\gamma(X, Y) &= a_d X^d + a_{d-1} X^{d-1} Y + \cdots + a_0 Y^d, \\ G^\gamma(X, Y) &= b_d X^d + b_{d-1} X^{d-1} Y + \cdots + b_0 Y^d. \end{aligned}$$

For each $\beta \in \mathcal{O}$, the map $\nu^\beta := \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \in \text{GL}_2(\mathcal{O})$ stabilizes ζ_G and takes the path $[0, \infty]$ to $[\beta, \infty]$. Put $\gamma^\beta = \gamma \circ \nu^\beta$; then $\gamma^\beta(\zeta_G) = Q$ and since $\varphi^{\gamma^\beta} = (\varphi^\gamma)^{\nu^\beta}$ it follows that the pair $(F^{\gamma^\beta}, G^{\gamma^\beta})$ given by

$$\begin{bmatrix} F^{\gamma^\beta}(X, Y) \\ G^{\gamma^\beta}(X, Y) \end{bmatrix} = \text{Adj}(\nu^\beta) \circ \begin{bmatrix} F^\gamma \\ G^\gamma \end{bmatrix} \circ \nu^\beta \circ \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F^\gamma(X + \beta Y, Y) - \beta G^\gamma(X + \beta Y, Y) \\ G^\gamma(X + \beta Y, Y) \end{bmatrix}$$

is another representation of φ at Q . It is normalized since $\nu^\beta \in \text{GL}_2(\mathcal{O})$. Write

$$(16) \quad \begin{aligned} F^{\gamma^\beta}(X, Y) &= a_d(\beta) X^d + a_{d-1}(\beta) X^{d-1} Y + \cdots + a_0(\beta) Y^d, \\ G^{\gamma^\beta}(X, Y) &= b_d(\beta) X^d + b_{d-1}(\beta) X^{d-1} Y + \cdots + b_0(\beta) Y^d. \end{aligned}$$

Lemma 1.4. *Let Q be a type II point; suppose $Q = \gamma(\zeta_G)$ where $\gamma \in \text{GL}_2(K)$ is such that (F^γ, G^γ) is normalized. Then for each direction $\vec{v} \in T_Q$*

(A) $\text{ordRes}_\varphi(\cdot)$ is locally decreasing in the direction \vec{v} if and only if

$$\vec{v} = \vec{v}_{Q, \gamma(\infty)} \quad \text{and} \quad \begin{cases} \text{ord}(a_\ell) > 0 \text{ when } (d+1)/2 \leq \ell \leq d \text{ and} \\ \text{ord}(b_\ell) > 0 \text{ when } (d-1)/2 \leq \ell \leq d, \end{cases}$$

or for some $\beta \in \mathcal{O}$,

$$\vec{v} = \vec{v}_{Q, \gamma(\beta)} \quad \text{and} \quad \begin{cases} \text{ord}(a_\ell(\beta)) > 0 \text{ when } 0 \leq \ell \leq (d+1)/2 \text{ and} \\ \text{ord}(b_\ell(\beta)) > 0 \text{ when } 0 \leq \ell \leq (d-1)/2. \end{cases}$$

(B) $\text{ordRes}_\varphi(\cdot)$ is locally constant in the direction \vec{v} if and only if d is odd and

$$\vec{v} = \vec{v}_{Q, \gamma(\infty)} \text{ and } \begin{cases} \text{ord}(a_{(d+1)/2}) = 0 \text{ or } \text{ord}(b_{(d-1)/2}) = 0, \text{ and} \\ \text{ord}(a_\ell) > 0 \text{ when } (d+1)/2 < \ell \leq d, \text{ and} \\ \text{ord}(b_\ell) > 0 \text{ when } (d-1)/2 < \ell \leq d \end{cases},$$

or d is odd and for some $\beta \in \mathcal{O}$,

$$\vec{v} = \vec{v}_{Q, \gamma(\beta)} \text{ and } \begin{cases} \text{ord}(a_{(d+1)/2}(\beta)) = 0 \text{ or } \text{ord}(b_{(d-1)/2}(\beta)) = 0, \text{ and} \\ \text{ord}(a_\ell(\beta)) > 0 \text{ when } 0 \leq \ell < (d+1)/2, \text{ and} \\ \text{ord}(b_\ell(\beta)) > 0 \text{ when } 0 \leq \ell < (d-1)/2 \end{cases}.$$

(C) $\text{ordRes}_\varphi(\cdot)$ is increasing in the direction \vec{v} , otherwise.

Proof. Note that $\gamma([0, \zeta_G]) = [\gamma(0), Q]$ and $\gamma([\zeta_G, \infty]) = [Q, \gamma(\infty)]$. We will prove the criteria for the directions $\vec{v}_{\gamma(0)}$ and $\vec{v}_{\gamma(\infty)}$ using formula (13) and the normalized representation (F^γ, G^γ) . Since $\gamma^\beta([0, \zeta_G]) = [\gamma(\beta), Q]$, the criteria for the directions $\vec{v}_{\gamma(\beta)}$ with arbitrary $\beta \in \mathcal{O}$ follow by applying the same arguments to $(F^{\gamma^\beta}, G^{\gamma^\beta})$.

Using the same notation as in formulas (12) and (13), for each $A \in K^\times$ put $Q_A = \gamma(\zeta_{|A|}) = \gamma_A(\zeta_G)$. Making the constants C_ℓ, D_ℓ in formula (13) explicit, we have

$$(17) \quad \begin{aligned} \text{ordRes}_\varphi(Q_A) - \text{ordRes}_\varphi(Q) &= \max \left(\max_{0 \leq \ell \leq d} ((d^2 + d - 2d\ell)t - 2d \text{ord}(a_\ell)), \right. \\ &\quad \left. \max_{0 \leq \ell \leq d} ((d^2 + d - 2d(\ell+1))t - 2d \text{ord}(b_\ell)) \right), \end{aligned}$$

where $t = \text{ord}(A)$. By assumption some $\text{ord}(a_\ell)$ or $\text{ord}(b_\ell)$ is 0, and $\text{ord}(a_\ell), \text{ord}(b_\ell) \geq 0$ for each ℓ . When $t = 0$ we have $Q_A = Q$ and both sides of (17) are 0.

Values of $t > 0$ correspond to points in the direction $\vec{v}_{\gamma(0)}$ at Q . For small positive t , the right side of (17) will be negative if and only if each of the affine functions in (17) with a nonnegative slope has a negative constant term. Hence $\text{ordRes}_\varphi(\cdot)$ is locally decreasing in the direction $\vec{v}_{\gamma(0)} \in T_Q$ if and only if $\text{ord}(a_\ell) > 0$ for each ℓ such that $d^2 + d - 2d\ell \geq 0$, and $\text{ord}(b_\ell) > 0$ for each ℓ such that $d^2 + d - 2d(\ell+1) \geq 0$. Similarly $\text{ordRes}_\varphi(\cdot)$ is locally constant in the direction $\vec{v}_{\gamma(0)}$ if and only if one of the affine functions with slope 0 has a constant term 0, and each of the affine functions with positive slope has negative constant term. This happens if and only if d is odd, either $\text{ord}(a_{(d+1)/2}) = 0$ or $\text{ord}(b_{(d-1)/2}) = 0$, $\text{ord}(a_\ell) > 0$ for each ℓ such that $d^2 + d - 2d\ell > 0$, and $\text{ord}(b_\ell) > 0$ for each ℓ such that $d^2 + d - 2d(\ell+1) > 0$.

Values of $t < 0$ correspond to points in the direction $\vec{v}_{\gamma(\infty)}$ at Q . For small negative t , the right side of (17) will be negative if and only if each of the affine functions in (17) with a nonpositive slope has a negative constant term. Hence $\text{ordRes}_\varphi(\cdot)$ is locally decreasing in the direction $\vec{v}_{\gamma(\infty)}$ if and only if $\text{ord}(a_\ell) > 0$ for each ℓ such that $d^2 + d - 2d\ell \leq 0$, and $\text{ord}(b_\ell) > 0$ for each ℓ such that $d^2 + d - 2d(\ell+1) \leq 0$. Similarly $\text{ordRes}_\varphi(\cdot)$ is locally constant in the direction $\vec{v}_{\gamma(\infty)}$ if and only if d is odd, either $\text{ord}(a_{(d+1)/2}) = 0$ or $\text{ord}(b_{(d-1)/2}) = 0$, $\text{ord}(a_\ell) > 0$ for each ℓ such that $d^2 + d - 2d\ell < 0$, and $\text{ord}(b_\ell) > 0$ for each ℓ such that $d^2 + d - 2d(\ell+1) < 0$. \square

Lemma 1.5. *If $d \geq 2$ is even, $\text{ordRes}_\varphi(\cdot)$ is never locally constant. If $d \geq 3$ is odd, then at each $Q \in \mathbb{P}_{\text{Berk}}^1$, there are at most two directions in T_Q where $\text{ordRes}_\varphi(\cdot)$ is locally constant.*

Proof. If $d \geq 2$ is even, then on any path the slope of each affine piece of $\text{ordRes}_\varphi(\cdot)$ is an integer $m \equiv d^2 + d \pmod{2d}$, hence is nonzero.

Suppose $d \geq 3$ is odd. If $Q \in \mathbb{P}_{\text{Berk}}^1$ is of type I, III, or IV then there are at most two directions in T_Q , so trivially there are at most two directions in T_Q in which $\text{ordRes}_\varphi(\cdot)$ is locally constant. Let Q be a type II point with at least two distinct directions where $\text{ordRes}_\varphi(\cdot)$ is locally constant, say \vec{v}_α and \vec{v}_β . Take any $\gamma \in \text{GL}_2(K)$ with $Q = \gamma(\zeta_G)$. After replacing γ with $\gamma\tau$ for a suitable $\tau \in \text{GL}_2(\mathcal{O})$, we can assume that $\vec{v}_\alpha = \vec{v}_{\gamma(0)}$ and $\vec{v}_\beta = \vec{v}_{\gamma(\infty)}$. Also, after replacing γ with $c\gamma$ for a suitable $c \in K^\times$, we can assume that (F^γ, G^γ) is a normalized representation of φ . Write $F^\gamma(X, Y) = a_d X^d + a_{d-1} X^{d-1} Y + \dots + a_0 Y^d$, $G^\gamma(X, Y) = b_d X^d + b_{d-1} X^{d-1} Y + \dots + b_0 Y^d$. By Lemma 1.4(B), if we put $D = (d+1)/2$ and $E = (d-1)/2$, then $\text{ord}(a_\ell) > 0$ for all $\ell \neq D$, $\text{ord}(b_\ell) > 0$ for all $\ell \neq E$, and either $\text{ord}(a_D) = 0$ or $\text{ord}(b_E) = 0$. Since $d \geq 3$, we have $D, E \geq 1$.

First suppose $\text{ord}(b_E) = 0$; then $G^\gamma(X, Y) \equiv b_E X^E Y^{d-E} \pmod{\mathfrak{M}}$, so for each $\beta \in \mathcal{O}$

$$G^{\gamma\beta}(X, Y) := G^\gamma(X + \beta Y, Y) \equiv b_E (X + \beta)^E Y^{d-E} \pmod{\mathfrak{M}}.$$

Comparing this with (16) shows $b_0(\beta) \equiv b_E \beta^E \pmod{\mathfrak{M}}$. If $\beta \not\equiv 0 \pmod{\mathfrak{M}}$, this means $\text{ord}(b_0(\beta)) = 0$, so the criterion in Lemma 1.4(B) is not met for the direction $\vec{v}_{\gamma(\beta)}$. Thus $\vec{v}_{\gamma(0)}$ and $\vec{v}_{\gamma(\infty)}$ are the only directions in which $\text{ordRes}_\varphi(\cdot)$ is locally constant.

Next suppose $\text{ord}(b_E) > 0$, so necessarily $\text{ord}(a_D) = 0$. Then $G^\gamma(X, Y) \equiv 0 \pmod{\mathfrak{M}}$ and $F^\gamma(X, Y) \equiv a_D X^D Y^{d-D} \pmod{\mathfrak{M}}$, so for each $\beta \in \mathcal{O}$

$$F^{\gamma\beta}(X, Y) := F^\gamma(X + \beta Y, Y) - \beta G^\gamma(X + \beta Y, Y) \equiv a_D (X + \beta)^D Y^{d-D} \pmod{\mathfrak{M}}.$$

Comparing this with (16) shows $a_0(\beta) \equiv a_D \beta^D \pmod{\mathfrak{M}}$. When $\beta \not\equiv 0 \pmod{\mathfrak{M}}$, this means $\text{ord}(a_0(\beta)) = 0$, so the criterion in Lemma 1.4(B) is not met for the direction $\vec{v}_{\gamma(\beta)}$, and again $\vec{v}_{\gamma(0)}$ and $\vec{v}_{\gamma(\infty)}$ are the only directions in which $\text{ordRes}_\varphi(\cdot)$ can be locally constant. \square

Remark. Using a similar argument, one can show that at any type II point there can be at most one direction in which $\text{ordRes}_\varphi(\cdot)$ is locally decreasing, without appealing to convexity.

Our next goal is to show that $\text{ordRes}_\varphi(\cdot)$ is strictly increasing as one moves away from the tree $\Gamma_{\text{Fix}, \varphi^{-1}(\infty)}$ in $\mathbb{P}_{\text{Berk}}^1$ spanned by the fixed points and the poles of φ . This means that $\text{ordRes}_\varphi(\cdot)$ achieves a minimum on $\mathbb{P}_{\text{Berk}}^1$, and shows that the locus $\text{MinResLoc}(\varphi)$ where it takes on its minimum is contained in that tree.

Two main facts underlie this. The first is that the group of affine transformations $\text{Aff}_2(K) = \{az + b : a \in K^\times, b \in K\}$, corresponding to matrices $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in \text{GL}_2(K)$, acts transitively on type II points. Indeed, if Q corresponds to a disc $D(b, r)$ with $r \in |K^\times|$, and $|a| = r$, then $\gamma(z) = az + b$ takes ζ_G to Q . The second is that the fixed points of φ are equivariant under $\text{GL}_2(K)$, and the poles are equivariant under $\text{Aff}_2(K)$: for each $\gamma \in \text{GL}_2(K)$, Δ is a fixed point of φ iff $\gamma^{-1}(\Delta)$ is a fixed point of φ^γ ; and for each $\gamma \in \text{Aff}_2(K)$, δ is a pole of φ iff $\gamma^{-1}(\delta)$ is a pole of φ^γ .

Lemma 1.6. *If $d \geq 2$, the set of poles and fixed points of φ in $\mathbb{P}^1(K)$ contains at least two distinct elements.*

Proof. The fixed points of φ correspond to solutions of $\varphi(z) = z$ in $\mathbb{P}^1(K)$. Using the representation $(F(X, Y), G(X, Y))$ for $\varphi(z)$, we obtain the representation $(YF(X, Y) - XG(X, Y), G(X, Y))$ for $\varphi(z) - z$.

Suppose all the poles and fixed points of φ occur at a single point $\alpha \in \mathbb{P}^1(K)$. If $\alpha = \infty$, there are $C, D \in K^\times$ such that $G(X, Y) = CY^d$ and $YF(X, Y) - XG(X, Y) = DY^{d+1}$. Solving, we see that $YF(X, Y) = DY^{d+1} + CXY^d$. Since $d \geq 2$, this contradicts that $F(X, Y)$ and $G(X, Y)$ have no common factors. If $\alpha \in K$, there are $C, D \in K^\times$ such that $G(X, Y) = C(X - \alpha Y)^d$ and $YF(X, Y) - XG(X, Y) = D(X - \alpha Y)^{d+1}$. In this case $YF(X, Y) = D(X - \alpha Y)^{d+1} + CX(X - \alpha Y)^d$, which again contradicts that $F(X, Y)$ and $G(X, Y)$ have no common factors. \square

Proposition 1.7. *If $d \geq 2$, the function $\text{ordRes}_\varphi(\cdot)$ is strictly increasing as one moves away from the tree $\Gamma_{\text{Fix}, \varphi^{-1}(\infty)}$ in $\mathbb{P}_{\text{Berk}}^1$ spanned by the fixed points and poles of φ in $\mathbb{P}^1(K)$.*

Proof. Let $\Gamma = \Gamma_{\text{Fix}, \infty}(\varphi)$ be the tree spanned by the fixed points and poles of φ . Branches off Γ in $\mathbb{P}_{\text{Berk}}^1$ can only occur at type II points. By the convexity of $\text{ordRes}_\varphi(\cdot)$, it suffices to show that at each type II point $Q \in \Gamma$, $\text{ordRes}_\varphi(\cdot)$ is increasing in each direction $\vec{v} \in T_Q$ which points away from Γ .

Fix a type II point $Q \in \Gamma$, and let $\vec{v} \in T_Q$ be a direction away from Γ . Let $\gamma \in \text{Aff}_2(K)$ be such that $\gamma(\zeta_G) = Q$. If $\vec{v} = \vec{v}_{Q, \infty}$, then $\gamma_*(\vec{v}_{\zeta_G, \infty}) = \vec{v}$. If $\vec{v} \neq \vec{v}_{Q, \infty}$, there is some $\beta \in \mathcal{O}$ such that $\gamma_*(\vec{v}_{\zeta_G, \beta}) = \vec{v}$, and after replacing γ with $\gamma^\beta = \gamma \circ \nu^\beta$ we can assume that $\gamma_*(\vec{v}_{\zeta_G, 0}) = \vec{v}$. Finally, by replacing γ with $c\gamma$ for some $c \in K^\times$, we can assume that the representation (F^γ, G^γ) of φ^γ is normalized.

First suppose $\vec{v} = \gamma_*(\vec{v}_{\zeta_G, \infty}) = \vec{v}_{\gamma(\infty)}$. By the equivariance of poles and fixed points under $\text{Aff}_2(K)$, φ^γ has no poles or fixed points in the direction \vec{v}_∞ at ζ_G . As in (10), write $F^\gamma(X, Y) = a_d X^d + a_{d-1} X^{d-1} Y + \cdots + a_0 Y^d$, $G^\gamma(X, Y) = b_d X^d + b_{d-1} X^{d-1} Y + \cdots + b_0 Y^d$. By hypothesis the poles δ_i of φ^γ all belong to \mathcal{O} , so we can factor $G^\gamma(X, Y) = b_d \cdot \prod_{i=1}^d (X - \delta_i Y)$ where $|\delta_i| \leq 1$ for each i . Expanding this and comparing coefficients shows that $\max(|b_d|, \dots, |b_0|) = |b_d|$. Likewise, the fixed points Δ_i of φ^γ all belong to \mathcal{O} . Since the fixed points are the zeros of

$$YF^\gamma(X, Y) - XG^\gamma(X, Y) = a_d X^{d+1} + (a_{d-1} - b_d) X^d Y + \cdots + (a_0 - b_1) X Y^d - b_0 Y^{d+1},$$

we can write $XF^\gamma(X, Y) - XG^\gamma(X, Y) = a_d \prod_{i=1}^{d+1} (X - \Delta_i Y)$. Expanding this and comparing coefficients shows that $\max(|a_d|, |a_{d-1} - b_d|, \dots, |a_0 - b_1|, |b_0|) = |a_d|$. However, it is an easy consequence of the ultrametric inequality that

$$(18) \quad \begin{aligned} & \max(|a_d|, |a_{d-1} - b_d|, \dots, |a_0 - b_1|, |b_0|, |b_d|, |b_{d-1}|, \dots, |b_0|) \\ &= \max(|a_d|, |a_{d-1}|, \dots, |a_0|, |b_d|, |b_{d-1}|, \dots, |b_0|). \end{aligned}$$

Thus $\max(|a_d|, |a_{d-1}|, \dots, |a_0|, |b_d|, |b_{d-1}|, \dots, |b_0|) = \max(|a_d|, |b_d|)$. Since (F^γ, G^γ) is normalized, it follows that $\text{ord}(a_d) = 0$ or $\text{ord}(b_d) = 0$. By Lemma 1.4, $\text{ordRes}_\varphi(\cdot)$ cannot be decreasing or constant in the direction $\vec{v} = \vec{v}_{\gamma(\infty)}$, so it must be increasing.

Next suppose $\vec{v} = \gamma_*(\vec{v}_{\zeta_G, 0}) = \vec{v}_{\gamma(0)}$. In this case φ^γ has no poles or fixed points in the direction \vec{v}_0 at ζ_G . As before, write $F^\gamma(X, Y) = a_d X^d + a_{d-1} X^{d-1} Y + \cdots + a_0 Y^d$, $G^\gamma(X, Y) = b_d X^d + b_{d-1} X^{d-1} Y + \cdots + b_0 Y^d$. By hypothesis the poles of φ^γ belong to $(K \setminus \mathfrak{M}) \cup \{\infty\}$, so we can factor $G^\gamma(X, Y) = CY^m \cdot \prod_{i=1}^{d-m} (X - \delta_i Y)$ for some $C \in K^\times$, where m is the number of poles of φ^γ at ∞ and $|\delta_i| \geq 1$ for $i = 1, \dots, d - m$.

Expanding and comparing coefficients shows that $|b_0| = \max(|b_d|, \dots, |b_0|)$. Likewise, the fixed points of φ^γ all belong to $(K \setminus \mathfrak{M}) \cup \{\infty\}$, so for some $D \in K^\times$ we can write $XF^\gamma(X, Y) - XG^\gamma(X, Y) = D \cdot Y^n \prod_{i=1}^{d-n} (X - \Delta_i Y)$ where n is the number of fixed points of φ^γ at ∞ , and $|\Delta_i| \geq 1$ for $i = 1, \dots, d-n$. Expanding and comparing coefficients shows that $|b_0| = \max(|a_d|, |a_{d-1} - b_d|, \dots, |a_0 - b_1|, |b_0|)$. Using (18) we see that $|b_0| = \max(|a_d|, |a_{d-1}|, \dots, |a_0|, |b_d|, |b_{d-1}|, \dots, |b_0|)$. Since (F^γ, G^γ) is normalized, it must be that $\text{ord}(b_0) = 0$. By Lemma 1.4, $\text{ordRes}_\varphi(\cdot)$ cannot be locally decreasing or constant in the direction $\vec{v} = \vec{v}_{\gamma(0)}$, so it must be increasing. \square

Proposition 1.8. *Suppose $d \geq 2$. Given a point $x \in \mathbb{P}^1(K)$, let ξ be the unique point in $[\zeta_G, x]$ such that $\rho(\zeta_G, x) = \frac{2}{d-1} \text{ordRes}(\varphi)$. Then $\text{ordRes}_\varphi(\cdot)$ is increasing along $[\xi, x]$ as one moves away from ξ .*

Proof. Choose a point $y \in \mathbb{P}^1(K)$ lying in a different direction at ζ_G than x . Then there is a $\gamma \in \text{GL}_2(\mathcal{O})$ such that $\gamma(0) = x$, $\gamma(\infty) = y$, and $\gamma([0, \infty]) = [x, y]$. Since $\gamma \in \text{GL}_2(\mathcal{O})$, it fixes ζ_G . Thus $\text{ordRes}_{\varphi^\gamma}(\zeta_G) = \text{ordRes}(\varphi^\gamma) = \text{ordRes}(\varphi)$. Let (F^γ, G^γ) be a representation of φ^γ , as in (10); after scaling (F^γ, G^γ) we can assume it is normalized. At least one of the coefficients a_0, b_0 in F^γ, G^γ must be nonzero. Expanding the determinant (1) for $\text{Res}(F^\gamma, G^\gamma)$ using its last column, one sees that $\min(\text{ord}(a_0), \text{ord}(b_0)) \leq \text{ordRes}(\varphi)$. Similarly, $\min(\text{ord}(a_d), \text{ord}(b_d)) \leq \text{ordRes}(\varphi)$.

Given $A \in K^\times$, put $Q_A = \zeta_{D(0, |A|)}$ and let $(F^{\gamma_A}, G^{\gamma_A})$ be as in (11). By (12), (13) and the discussion above,

$$\begin{aligned}
 \text{ordRes}_{\varphi^\gamma}(Q_A) - \text{ordRes}(\varphi) &= (d^2 + d)\text{ord}(A) \\
 &\quad - 2d \min(\text{ord}(a_0), \dots, \text{ord}(A^d a_d), \text{ord}(A b_0), \dots, \text{ord}(A^{d+1} b_d)) \\
 &\geq \max(-2d \text{ord}(a_0) + (d^2 + d)\text{ord}(A), -2d \text{ord}(b_0) + (d^2 - d)\text{ord}(A), \\
 &\quad -2d \text{ord}(a_d) + (d - d^2)\text{ord}(A), -2d \text{ord}(b_d) + (-d - d^2)\text{ord}(A)) \\
 (19) \quad &\geq -2d \text{ordRes}(\varphi) + \max((d^2 - d)\text{ord}(A), (d - d^2)\text{ord}(A)).
 \end{aligned}$$

Since $\text{ordRes}_{\varphi^\gamma}(\zeta_G) - \text{ordRes}(\varphi) = 0$, the minimum value of $\text{ordRes}_{\varphi^\gamma}(\cdot) - \text{ordRes}(\varphi)$ on $[0, \infty]$ is nonpositive. Since $d \geq 2$, we have $d^2 - d > 0$; hence the right side of (19) is nonpositive precisely when

$$(20) \quad -\frac{2}{d-1} \text{ordRes}(\varphi) \leq \text{ord}(A) \leq \frac{2}{d-1} \text{ordRes}(\varphi).$$

By convexity, $\text{ordRes}_{\varphi^\gamma}(Q_A)$ must be increasing with $\text{ord}(A)$ for $\text{ord}(A) > \frac{2}{d-1} \text{ordRes}(\varphi)$. Since $\text{ordRes}_\varphi(\gamma(Q_A)) = \text{ordRes}_{\varphi^\gamma}(Q_A)$, the Proposition follows. \square

Proof of Theorem 0.1. Assume $d \geq 2$. By Proposition 1.3, the function $\text{ordRes}_\varphi(\cdot)$ on type II points extends to a function $\text{ordRes}_\varphi : \mathbb{P}_{\text{Berk}}^1 \rightarrow [0, \infty]$ which is continuous with respect to the strong topology, finite on \mathbb{H}_{Berk} and ∞ on $\mathbb{P}^1(K)$, and piecewise affine and convex upwards with respect to $\rho(x, y)$ on each path. By Lemma 1.6, the tree $\Gamma = \Gamma_{\text{Fix}, \infty}(\varphi)$ spanned by the poles and fixed points of φ is nontrivial, and by Proposition 1.7, $\text{ordRes}_\varphi(\cdot)$ is strictly increasing as one moves away from Γ . It follows that $\text{ordRes}_\varphi(\cdot)$ takes on a minimum value on $\mathbb{P}_{\text{Berk}}^1$, and that the set $\text{MinResLoc}(\varphi)$ where the minimum is achieved is a compact connected subset of $\Gamma \cap \mathbb{H}_{\text{Berk}}$.

On any path the slopes of $\text{ordRes}_\varphi(\cdot)$ are integers $m \equiv d^2 + d \pmod{2d}$. If d is even, then $d^2 + d \equiv d \pmod{2d}$, so none of the slopes are 0. Since the breaks between affine

pieces occur at type II points, $\text{MinResLoc}(\varphi)$ consists of a single type II point. If d is odd, then $d^2 + d \equiv 0 \pmod{2d}$. By Lemma 1.5, at each $Q \in \text{MinResLoc}(\varphi)$ there are at most two directions where $\text{ordRes}_\varphi(\cdot)$ is constant. Thus $\text{MinResLoc}(\varphi)$ is either a single type II point, or a segment with type II endpoints.

We next show that $\text{MinResLoc}(\varphi) \subseteq \{z \in \mathbb{H}_{\text{Berk}} : \rho(\zeta_G, z) \leq \frac{2}{d-1} \text{ordRes}(\varphi)\}$. Fix z with $\rho(\zeta_G, z) > \frac{2}{d-1} \text{ordRes}(\varphi)$. Let ξ be the unique point on $[\zeta_G, z]$ with $\rho(\zeta_G, \xi) = \frac{2}{d-1} \text{ordRes}(\varphi)$; then ξ is of type II. Let $x \in \mathbb{P}^1(K)$ be a type I point whose direction from ξ is the same as that of z . By Proposition 1.8, $\text{ordRes}_\varphi(\cdot)$ is increasing along $[\xi, x]$. Since $[\xi, z]$ and $[\xi, x]$ share an initial segment, by convexity $\text{ordRes}_\varphi(\cdot)$ is increasing along $[\xi, z]$. Thus $z \notin \text{MinResLoc}(\varphi)$.

The final assertion in Theorem 0.1 reformulates a result of Favre and Rivera-Letelier ([14], Theorem E). Suppose the minimal value of $\text{ordRes}_\varphi(\xi)$ is 0. By what has been shown above, there is a type II point $\xi \in \text{MinResLoc}(\varphi)$ where $\text{ordRes}_\varphi(\xi) = 0$. Let $\gamma \in \text{GL}_2(K)$ be such that $\gamma(\zeta_G) = \xi$. Then $\text{ordRes}(\varphi^\gamma) = 0$, so φ^γ has good reduction. Since $d \geq 2$, by ([14], Theorem E, or [2], Proposition 10.5), ξ is the unique point where φ achieves good reduction. Thus $\text{MinResLoc}(\varphi) = \{\xi\}$. \square

Proof of Theorem 0.2. Since $\text{ordRes}_\varphi(\cdot)$ and $\text{ordRes}_{\tilde{\varphi}}(\cdot)$ are continuous for the strong topology, to prove the first assertion it suffices to show that if (4) holds then $\text{ordRes}_\varphi(\xi) = \text{ordRes}_{\tilde{\varphi}}(\xi)$ for all type II points ξ with $\rho(\zeta_G, \xi) \leq M$. If $\xi = \zeta_{D(a,r)}$ then the path from ζ_G to ξ goes from ζ_G to $\zeta_{D(0,T)}$ where $T = \max(1, |a|)$, and then from $\zeta_{D(0,T)} = \zeta_{D(a,T)}$ to ξ . Hence if $A, B \in K^\times$ are such that $|A| = 1/T$ and $|B| = r/T$, then $\rho(\zeta_G, \xi) = \text{ord}(A \cdot B)$ and $\xi = \gamma(\zeta_G)$, where

$$\gamma = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} \cdot \begin{bmatrix} B & aA \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} B & aA \\ 0 & A \end{bmatrix} \in \text{GL}_2(K) \cap M_2(\mathcal{O}).$$

By (8) we have

$$(21) \quad \text{ordRes}_\varphi(\xi) = \text{ordRes}(F, G) + (d^2 + d)(\text{ord}(A \cdot B)) - 2d \min(\text{ord}(F^\gamma), \text{ord}(G^\gamma))$$

where F^γ and G^γ are given by (7), and an analogous formula holds for $\text{ordRes}_{\tilde{\varphi}}(\xi)$. Since $\text{ordRes}_\varphi(\xi) \geq 0$, $\text{ordRes}(F, G) = R$, and $\text{ord}(A \cdot B) \leq M$, we conclude from (21) that

$$\min(\text{ord}(F^\gamma), \text{ord}(G^\gamma)) \leq \frac{1}{2d}(R + (d^2 + d)M).$$

Now (4) gives $\min(\text{ord}(F^\gamma), \text{ord}(G^\gamma)) = \min(\text{ord}(\tilde{F}^\gamma), \text{ord}(\tilde{G}^\gamma))$ and $\text{ordRes}(F, G) = \text{ordRes}(\tilde{F}, \tilde{G}) = R$. Hence $\text{ordRes}_\varphi(\xi) = \text{ordRes}_{\tilde{\varphi}}(\xi)$.

The second assertion follows by taking $M = \frac{2}{d-1}R + \varepsilon$ in (4), with $\varepsilon > 0$ small, and using Theorem 0.1. \square

2. EXAMPLES

Throughout this section, we write \mathbb{C}_p for the completion of the algebraic closure of \mathbb{Q}_p . The valuation $\text{ord}(\cdot)$ on \mathbb{C}_p will be normalized so that $\text{ord}(p) = 1$, and $|\cdot|_p = p^{-\text{ord}(\cdot)}$ is the usual absolute value on \mathbb{C}_p . We write $\text{Res}(\varphi)$ for $\text{Res}(F, G)$, where (F, G) is the obvious homogenization of the pair of polynomials defining φ .

We first give two examples where $\varphi(z)$ has potential good reduction.

Example 2.1. The function $\varphi(z) = \frac{z^d - p}{z^{d-1}}$, with p arbitrary and $K = \mathbb{C}_p$.

Here $\text{Res}(\varphi) = (-1)^{d(d-1)/2} p^{d-1}$, so $\text{ordRes}(\varphi) = d - 1$. The poles of φ are 0 and ∞ , and there is a $(d + 1)$ -fold fixed point at ∞ . The tree Γ spanned the fixed points and poles is just the path $[0, \infty]$.

Consider $\text{ordRes}_\varphi(\cdot)$ on $[0, \infty]$. Let $Q_A \in \mathbb{P}_{\text{Berk}}^1$ correspond to $D(0, |A|)$; by (13)

$$\begin{aligned} \text{ordRes}_\varphi(Q_A) &= (d - 1) + (d^2 + d)\text{ord}(A) - 2d \min(\text{ord}(A^d), \text{ord}(p), \text{ord}(A \cdot A^{d-1})) \\ &= \max((d - 1) + (d - d^2)\text{ord}(A), (-d - 1) + (d^2 + d)\text{ord}(A)). \end{aligned}$$

This achieves its minimum when $\text{ord}(A) = 1/d$, and $\text{ordRes}_\varphi(\zeta_{D(0, p^{1/d})}) = 0$.

Thus $\varphi(z)$ has potential good reduction at the point $\zeta_{D(0, p^{1/d})}$, and conjugation by $\gamma = \begin{bmatrix} p^{1/d} & 0 \\ 0 & 1 \end{bmatrix}$ achieves the necessary change of coordinates: indeed

$$\varphi^\gamma(z) = \frac{z^d - 1}{z^{d-1}}.$$

Here $\rho(\zeta_G, \zeta_{D(0, p^{1/d})}) = 1/d < \frac{2}{d-1} \text{ordRes}(\varphi) = 2$. Note also that $\text{ordRes}_\varphi(\zeta_G) = d - 1$, and $\text{ordRes}_\varphi(\zeta_{D(0, p)}) = d^2 - 1$.

Example 2.2. The function $\varphi(z) = \frac{z^2 - 1}{2z}$, with $K = \mathbb{C}_2$.

Here $\text{Res}(\varphi) = -4$, so $\text{ordRes}(\varphi) = 2$. The poles of φ are 0 and ∞ , and the fixed points are ∞ and $\pm i$, where $i = \sqrt{-1}$. If we write $\zeta_{D(a, r)}$ for the point in $\mathbb{P}_{\text{Berk}}^1$ corresponding to the disc $D(a, r) \subset K$, then the tree Γ spanned by $\{0, \infty, i, -i\}$ has branch points at $\zeta_G = \zeta_{D(0, 1)}$ and $\zeta_{D(i, 1/2)}$.

First consider $\text{ordRes}_\varphi(\cdot)$ on the path $[0, \infty]$. Let $Q_A = \zeta_{D(0, |A|)} \in \mathbb{P}_{\text{Berk}}^1$; then

$$\text{ordRes}_\varphi(Q_A) = \max(2 - 2\text{ord}(A), 2 + 6\text{ord}(A)).$$

This takes on its minimum when $\text{ord}(A) = 0$, where $\text{ordRes}_\varphi(Q_A) = 2$ and $Q_A = \zeta_G$.

Next consider $\text{ordRes}_\varphi(\cdot)$ on the path $[i, \infty]$. Let $\gamma = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}$, so $\gamma(0) = i$ and $\gamma(\infty) = \infty$. Then

$$\varphi^\gamma(z) = \frac{(z + i)^2 - 1}{2(z + i)} - i = \frac{z^2 - 4iz}{2z + 2i}.$$

Let Q_A be the point corresponding to the disc $D(i, |A|)$; then

$$\text{ordRes}_\varphi(Q_A) = \max(2 - 2\text{ord}(A), -2 + 2\text{ord}(A)).$$

This achieves its minimum when $\text{ord}(A) = 1$, and $\text{ordRes}_\varphi(Q_A) = 0$. The corresponding point Q_A is $\zeta_{D(i, 1/2)}$; note that $\rho(\zeta_G, \zeta_{D(i, 1/2)}) = 1 < \frac{2}{d-1} \text{ordRes}(\varphi) = 4$.

Thus $\varphi(z)$ has potential good reduction at $\zeta_{D(i, 1/2)}$, and the map $\eta = \gamma \circ \nu^2 = \begin{bmatrix} 2 & i \\ 0 & 1 \end{bmatrix}$ achieves the necessary change of coordinates. One sees that

$$\varphi^\eta(z) = \frac{z^2 - 2iz}{2z + i}$$

indeed has good reduction. The nearest point to $\zeta_{D(i,1/2)}$ in the tree spanned by $\mathbb{P}^1(\mathbb{Q}_2)$ is $\zeta_{D(1,1/\sqrt{2})}$; one sees that $\text{ordRes}_\varphi(\zeta_{D(1,1/\sqrt{2})}) = 1$. The nearest points in that tree with radii belonging to the value group $|\mathbb{Q}_2^\times|$ (we call such points \mathbb{Q}_2 -rational type II points) are ζ_G and $\zeta_{D(1,1/2)}$; one has $\text{ordRes}_\varphi(\zeta_G) = 2$ and $\text{ordRes}_\varphi(\zeta_{D(1,1/2)}) = 4$.

We next give several examples where φ does not have potential good reduction.

Example 2.3. The function $\varphi(z) = \frac{z^p - z}{p}$, with $K = \mathbb{C}_p$ for an arbitrary prime p .

It is known (see [2], Example 10.120) that the Berkovich Julia set of $\varphi(z)$ is contained in $\mathbb{P}^1(\mathbb{C}_p)$ (indeed, it is precisely \mathbb{Z}_p), and its invariant measure μ_φ is the additive Haar measure on \mathbb{Z}_p . Thus, $\varphi(z)$ cannot have potential good reduction; if it did, its Berkovich Julia set would be the unique point $Q \in \mathbb{H}_{\text{Berk}}$ where it attained good reduction. Below we will give a direct proof that φ does not have potential good reduction.

Here $d = p$, and $\text{Res}(\varphi) = p^p$, so $\text{ordRes}(\varphi) = p$. The fixed points of $\varphi(z)$ are ∞ and the solutions u_0, u_1, \dots, u_{p-1} to $z^p - (1+p)z = 0$. Since $z^p - (1+p)z \equiv z^p - z \equiv z(z-1)\cdots(z-(p-1)) \pmod{p}$, Hensel's Lemma shows that each u_i belongs to \mathbb{Z}_p , and we can label the u_i so that $u_0 = 0$ and $u_i \equiv i \pmod{p}$ for $i = 1, \dots, p-1$. The poles of $\varphi(z)$ are all at ∞ . The tree Γ spanned by $\{\infty, 0, u_1, \dots, u_{p-1}\}$ has ζ_G as its only branch point.

First consider $\text{ordRes}_\varphi(\cdot)$ on the path $[0, \infty]$. As before, write Q_A for $\zeta_{D(0,|A|)}$; by (13)

$$\begin{aligned} \text{ordRes}_\varphi(Q_A) &= p + (p^2 + p)\text{ord}(A) - 2p \min(\text{ord}(A^p), \text{ord}(A), \text{ord}(pA)) \\ &= \max(p + (p - p^2)\text{ord}(A), p + (p^2 - p)\text{ord}(A)) . \end{aligned}$$

The minimum is achieved when $\text{ord}(A) = 0$, corresponding to $\text{ordRes}_\varphi(\zeta_G) = p$.

Next fix i with $1 \leq i \leq p-1$, and consider $\text{ordRes}_\varphi(\cdot)$ on the path $[u_i, \infty]$. Taking $\gamma = \begin{bmatrix} 1 & u_i \\ 0 & 1 \end{bmatrix}$, we see that

$$\varphi^\gamma(z) = \frac{(z + u_i)^p - (z + u_i) - pu_i}{p} = \frac{a_p z^p + a_{p-1} z^{p-1} + \cdots + a_1 z}{p}$$

where $a_p = 1$, $a_j = \binom{p}{j} u_i^{p-j}$ for $j = 2, \dots, p-1$, and $a_1 = pu_i^{p-1} - 1$. In particular $\text{ord}(a_p) = \text{ord}(a_1) = 0$, and $\text{ord}(a_j) = 1$ for $j = 2, \dots, p-1$. By (13)

$$\begin{aligned} \text{ordRes}_\varphi(\gamma(Q_A)) &= \text{ordRes}_{\varphi^\gamma}(Q_A) \\ &= p + (p^2 + p)\text{ord}(A) - 2p \min(\text{ord}(a_p A^p), \dots, \text{ord}(a_1 A), \text{ord}(pA)) \\ &= \max(p + (p - p^2)\text{ord}(A), p + (p^2 - p)\text{ord}(A)) . \end{aligned}$$

Again the minimum is achieved when $\text{ord}(A) = 0$, corresponding to $\text{ordRes}_\varphi(\zeta_G) = p$. Thus $\text{MinResLoc}(\varphi) = \{\zeta_G\}$, and $\varphi(z)$ does not have potential good reduction. Here $\rho(\zeta_G, \zeta_G) = 0 < \frac{2}{d-1} \text{ordRes}(\varphi) = 2p/(p-1)$. Note that $\varphi(\zeta_G) = \zeta_{D(0,p)}$, so $\text{MinResLoc}(\varphi)$ is not fixed by φ .

Example 2.4. The function $\varphi(z) = \frac{p^4 z^3 + pz + 1}{p^6 z^3}$, where $K = \mathbb{C}_p$, and p is odd.

Here $\text{Res}(\varphi) = p^{18}$. The function $\varphi(z)$ has a triple pole at 0, and its fixed points are the roots of $-p^6 z^4 + p^4 z^3 + pz + 1$. By the theory of Newton Polygons, if the fixed points

u_1, \dots, u_4 are ordered by increasing size, then $|u_1| = p$, $|u_2| = |u_3| = p^{3/2}$, and $|u_4| = p^2$. The tree Γ spanned by $\{0, u_1, u_2, u_3, u_4\}$ has branch points at $\zeta_{D(0,p)}$ and $\zeta_{D(0,p^{3/2})}$.

Consider $\text{ordRes}_\varphi(\cdot)$ on the path $[0, \infty]$; note that only the subsegment $[0, \zeta_{D(0,p^2)}]$ is contained in Γ . Let $Q_A \in \mathbb{P}_{\text{Berk}}^1$ be the point corresponding to $D(0, |A|)$. Then

$$\text{ordRes}_\varphi(Q_A) = \max(-18 - 12 \text{ord}(A), -6 - 6 \text{ord}(A), 12 + 6 \text{ord}(A), 18 + 12 \text{ord}(A)) .$$

This function achieves its minimum value of 3 when $\text{ord}(A) = -3/2$; it has breaks when $\text{ord}(A) = -1$, $\text{ord}(A) = -3/2$, and $\text{ord}(A) = -2$.

The initial segments of $[\zeta_{D(0,p^{3/2})}, u_1]$ and $[\zeta_{D(0,p^{3/2})}, u_4]$ belong to $[0, \infty]$, so $\text{ordRes}_\varphi(\cdot)$ is increasing along them. To show that $\text{ordRes}_\varphi(\cdot)$ achieves its minimum on $\mathbb{P}_{\text{Berk}}^1$ at $\zeta_{D(0,p^{3/2})}$, it suffices to check that it is increasing along $[\zeta_{D(0,p^{3/2})}, u_2]$ and $[\zeta_{D(0,p^{3/2})}, u_3]$.

Take $\gamma = \begin{bmatrix} p^{3/2} & 0 \\ 0 & 1 \end{bmatrix}$; conjugating φ by γ brings $\zeta_{D(0,p^{3/2})}$ to ζ_G . One finds that

$$\varphi^\gamma(z) = \frac{z^3 + z + p^{1/2}}{p^{1/2}z} .$$

The fixed points of φ^γ lie in the directions of 0, $\pm i$ and ∞ at ζ_G , where $i = \sqrt{-1}$; these correspond to the directions of u_1 , u_2 , u_3 and u_4 at $\zeta_{D(0,p^{3/2})}$, respectively. Since p is odd, the directions of $\pm i$ at ζ_G are distinct. Conjugating φ^γ by $\nu = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}$ yields

$$\varphi^{\nu\gamma}(z) = \frac{(1 - ip^{1/2})z^3 + (3 + i)z^2 + (-2 + 3ip^{1/2})z}{p^{1/2}(z + i)^3} .$$

Since $\text{ord}(-2 + 3ip^{1/2}) = 0$ when p is odd, it follows from Lemma 1.4 (or directly from formula (13)), that $\text{ordRes}_\varphi(\cdot)$ is increasing in the direction of u_2 at $\zeta_{D(0,p^{3/2})}$. A similar argument applies for u_3 .

Thus $\varphi(z)$ does not have potential good reduction: $\text{MinResLoc}(\varphi) = \{\zeta_{D(0,p^{3/2})}\}$, with $\text{ordRes}_\varphi(\zeta_{D(0,p^{3/2})}) = 3$. Here $\varphi(\zeta_{D(0,p^{3/2})}) = \zeta_{D(0,p)}$, so $\text{MinResLoc}(\varphi)$ is not fixed by φ . Note that $\rho(\zeta_G, \zeta_{D(0,p^{3/2})}) = 3/2 < \frac{2}{d-1} \text{ordRes}(\varphi) = 12$. Note also that $\text{ordRes}_\varphi(\zeta_{D(0,p)}) = \text{ordRes}_\varphi(\zeta_{D(0,p^2)}) = 6$.

Example 2.5. The function $\varphi(z) = \frac{p^n z^3 + z^2 - p^n z}{-p^n z^2 + z + p^n}$ with $n > 0$ and $K = \mathbb{C}_p$ for any p .

Here $\text{Res}(\varphi) = -4p^{4n}$, so $\text{ordRes}(\varphi) = 4n + 2\text{ord}(2)$. The poles of $\varphi(z)$ are $\alpha_\pm = (1 \pm \sqrt{1 + 4p^{2n}})/(-2p^n)$, where

$$\alpha_- = -p^n + p^{3n} + \dots, \quad \alpha_+ = 1/p^n + p^n - p^{3n} + \dots,$$

and the fixed points are 0 and ∞ . The tree Γ spanned by $\{0, \infty, \alpha_-, \alpha_+\}$ has branch points at $\zeta_{D(0,1/p^n)}$ and $\zeta_{D(0,p^n)}$.

First consider $\text{ordRes}_\varphi(\cdot)$ on the path $[0, \infty]$. Let $Q_A \in \mathbb{P}_{\text{Berk}}^1$ be the point corresponding to $D(0, |A|)$; then

$$\text{ordRes}_\varphi(Q_A) = 2\text{ord}(2) + \max(-2n - 6 \text{ord}(A), 4n, -2n + 6 \text{ord}(A)) .$$

This takes its minimum value of $4n + 2\text{ord}(2)$ for all A with $\text{ord}(A) \in [-n, n]$.

Note that the paths $[\zeta_{D(0,1/p^n)}, \alpha_+]$ and $[\zeta_{D(0,1/p^n)}, -p^n]$ share an initial segment. Take $\gamma = \begin{bmatrix} p^n & 0 \\ 0 & 1 \end{bmatrix}$, $\nu = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and put $\eta = \gamma \circ \nu = \begin{bmatrix} p^n & -p^n \\ 0 & 1 \end{bmatrix}$. Then η takes $[0, \infty]$ to the path $[-p^n, \infty]$, with $\eta(\zeta_G) = \zeta_{D(0,1/p^n)}$. One computes

$$\varphi^\eta(z) = (\varphi^\gamma)^\nu(z) = \frac{p^{2n}(z-1)^3 - p^{2n}(z-1)^2 + z^2 - z + 1}{-p^{2n}(z-1)^2 + z}.$$

If we write the numerator of φ^η as $a_3 z^3 + a_2 z^2 + a_1 z + a_0$, then $\text{ord}(a_2) = 0$, and it follows from (13) that $\text{ordRes}_{\varphi^\eta}(\cdot)$ is increasing in the direction \vec{v}_0 at ζ_G . This means $\text{ordRes}_\varphi(\cdot)$ is increasing in the direction \vec{v}_{α_+} at $\zeta_{D(0,1/p^n)}$. By a similar argument, one sees that $\text{ordRes}_\varphi(\cdot)$ is increasing in the direction \vec{v}_{α_-} at $\zeta_{D(0,p^n)}$.

Thus $\text{MinResLoc}(\varphi)$ is the segment $[\zeta_{D(0,1/p^n)}, \zeta_{D(0,p^n)}]$, and the minimal value of $\text{ordRes}_\varphi(\cdot)$ is $4n + 2\text{ord}(2)$; in particular $\varphi(z)$ does not have potential good reduction. Each point of $[\zeta_{D(0,1/p^n)}, \zeta_{D(0,p^n)}]$ is fixed by φ : $\zeta_{D(0,p^\alpha)}$ is an indifferent fixed point for $-n < \alpha < n$, and $\zeta_{D(0,1/p^n)}$ and $\zeta_{D(0,p^n)}$ are repelling fixed points of degree 2. In this case $\text{MinResLoc}(\varphi)$ is contained in $\{z \in \mathbb{H}_{\text{Berk}} : \rho(\zeta_G, z) \leq n\}$, while $\frac{2}{d-1}\text{ordRes}(\varphi) = \frac{8}{3}n + \frac{4}{3}\text{ord}(2)$.

For rationality considerations later, it will be useful to examine some conjugates of $\varphi(z)$. For each $\gamma \in \text{GL}_2(K)$, it is a formal consequence of the definitions that for all Q

$$(22) \quad \text{ordRes}_{\varphi^\gamma}(Q) = \text{ordRes}_\varphi(\gamma(Q)).$$

To show this, by continuity it is enough to check it for type II points. Suppose $Q = \tau(\zeta_G)$ for some $\tau \in \text{GL}_2(K)$. Then

$$\text{ordRes}_{\varphi^\gamma}(Q) = \text{ordRes}_{\varphi^\gamma}(\tau(\zeta_G)) = \text{ordRes}_\varphi(\gamma(\tau(\zeta_G))) = \text{ordRes}_\varphi(\gamma(Q)).$$

Take $u \in \mathbb{C}_p$ with $|u| = 1$, and let $\gamma_1 = \begin{bmatrix} 1 & u \\ -1 & u \end{bmatrix}$. One easily sees that

$$(23) \quad \varphi_1(z) := \varphi^{\gamma_1}(z) = \frac{-z^3 + (4p^n + 1)u^2 z}{(4p^n - 1)z^2 + u^2},$$

and that $\gamma_1(\zeta_{D(u,1/p^n)}) = \zeta_{D(0,p^n)}$ and $\gamma_1(\zeta_{D(-u,1/p^n)}) = \zeta_{D(0,1/p^n)}$. It follows that $\text{MinResLoc}(\varphi_1)$ is the segment $[\zeta_{D(-u,1/p^n)}, \zeta_{D(u,1/p^n)}]$. When p is odd, the midpoint of this segment is $\zeta_G = \zeta_{D(0,1)}$. When $p = 2$, its midpoint is $\zeta_{D(u,1/2)}$.

Next conjugate $\varphi_1(z)$ by $\gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & p^{1/2} \end{bmatrix}$. Then

$$(24) \quad \varphi_2(z) := (\varphi_1)^{\gamma_2}(z) = \frac{-z^3 + (4p^n + 1)u^2 p z}{(4p^n - 1)z^2 + p u^2},$$

and $\text{MinResLoc}(\varphi_2) = [\zeta_{D(-up^{1/2}, 1/p^{(n+1/2)})}, \zeta_{D(up^{1/2}, 1/p^{(n+1/2)})}]$. When p is odd, the midpoint of this segment is $\zeta_{D(0,p^{-1/2})}$. When $p = 2$, its midpoint is $\zeta_{D(u2^{1/2}, 2^{-3/2})}$.

Example 2.6. The function $\varphi(z) = \frac{z^2}{(1 + pz)^4}$, where $K = \mathbb{C}_p$ and $p \geq 5$. This function was studied by Favre and Rivera-Letelier ([14]; or see [2], Example 10.124), who showed that its Berkovich Julia set is the segment $[\zeta_G, \zeta_{D(0,p^2)}]$ and that its invariant measure μ_φ is the uniform measure of mass 1 on that segment (relative to the path distance). Here $\text{Res}(\varphi) = p^8$. The poles of φ are all at $z = -1/p$, and the fixed points of φ are $z = 0$

and the roots of $1 + (4p - 1)z + 6p^2z^2 + 4p^3z^3 + p^4z^4 = 0$. By the theory of Newton polygons, these roots can be labeled so that $|u_1| = 1$ and $|u_2| = |u_3| = |u_4| = p^{4/3}$. The tree Γ spanned by $\{0, -1/p, u_1, u_2, u_3, u_4\}$ has branch points at ζ_G , $\zeta_{D(0,p)}$, and $\zeta_{D(0,p^{4/3})}$.

On the path $[0, \infty]$, we have

$$\text{ordRes}_\varphi(\zeta_{D(0,|A|)}) = \max(-24 - 20 \text{ord}(A), 8 + 4 \text{ord}(A)) ,$$

which takes its minimum value of $8/3$ at $\text{ord}(A) = 4/3$. Conjugating by $\gamma = \begin{bmatrix} p^{-4/3} & 0 \\ 0 & 1 \end{bmatrix}$ gives

$$(25) \quad \varphi^\gamma(z) = \frac{z^2}{z^4 + 4p^{1/3}z^3 + 6p^{2/3}z^2 + 4pz + p^{4/3}} .$$

The fixed points u_2, u_3, u_4 lie in the directions $\vec{v}_{p^{4/3}}, \vec{v}_{\zeta_3 p^{4/3}}, \vec{v}_{\zeta_3^2 p^{4/3}}$ at $\zeta_{D(0,p^{4/3})}$, where ζ_3 is a primitive cube root of unity, and it is easily checked that $\text{ordRes}_\varphi(\cdot)$ is increasing in each of those directions. Thus $\text{MinResLoc}(\varphi) = \{\zeta_{D(0,p^{4/3})}\}$. Note that $\zeta_{D(0,p^{4/3})}$ is fixed by φ ; indeed, by (25), $\zeta_{D(0,p^{4/3})}$ is a repelling fixed point of φ of degree 2. Also note that $\rho(\zeta_G, \zeta_{D(0,p^{4/3})}) = 4/3 < \frac{2}{d-1} \text{ordRes}(\varphi) = 16/3$.

Example 2.7. The function $\varphi(z) = \frac{pz^3 + z^2}{p}$, with $K = \mathbb{C}_p$ for an arbitrary prime p .

Here $\text{Res}(\varphi) = p^6$. The fixed points of $\varphi(z)$ are $0, \infty$ and the solutions u_1, u_2 to $pz^2 + z - p = 0$:

$$u_1 = p + p^3 + \cdots, \quad u_2 = -p^{-1} - p - p^3 + \cdots$$

so that $|u_1| = 1/p$, $|u_2| = p$. The poles of $\varphi(z)$ are all at ∞ . The tree Γ spanned by $\{0, \infty, u_1, u_2\}$ has branch points at $\zeta_{D(0,1/p)}$ and $\zeta_{D(0,p)}$.

First consider $\text{ordRes}_\varphi(\cdot)$ on $[0, \infty]$. We have

$$\begin{aligned} \text{ordRes}_\varphi(\zeta_{D(0,|A|)}) &= 6 + 12\text{ord}(A) - 6 \min(\text{ord}(pA^3), \text{ord}(A^2), \text{ord}(pA)) \\ &= \max(-6\text{ord}(A), 6, 6 + \text{ord}(A)) . \end{aligned}$$

This takes the constant value 6 when $-1 \leq \text{ord}(A) \leq 0$. By convexity, the minimum value of $\text{ordRes}_\varphi(\cdot)$ on $\mathbb{P}_{\text{Berk}}^1$ is 6, and $\text{MinResLoc}(\varphi)$ contains the segment $[\zeta_G, \zeta_{D(0,p)}]$.

To see that $\text{MinResLoc}(\varphi)$ contains no other points, note that the path $[\zeta_{D(0,p)}, u_2]$ shares an initial segment with $[\zeta_{D(0,p)}, p^{-1}]$. Conjugating φ by $\gamma = \begin{bmatrix} 1/p & 1/p \\ 0 & 1 \end{bmatrix}$, which takes 0 to p^{-1} and ζ_G to $\zeta_{D(0,1/p)}$, yields $\varphi^\gamma(z) = (z^3 + 4z^2 + 5z + (2 - p^2))/p^2$. One computes

$$\begin{aligned} \text{ordRes}_{\varphi^\gamma}(\zeta_{D(0,|A|)}) \\ = \max(6 - 6\text{ord}(A), 6 - 6\text{ord}(5) + 6\text{ord}(A), 6 - 6\text{ord}(2) + 12\text{ord}(A)) . \end{aligned}$$

Since either $\text{ord}(5) = 0$ or $\text{ord}(2) = 0$, the right side is increasing for small positive values of $\text{ord}(A)$. Thus $\text{ordRes}_\varphi(\cdot)$ is increasing along $[\zeta_{D(0,p)}, u_2]$, and $\text{MinResLoc}(\varphi) = [\zeta_G, \zeta_{D(0,p)}]$. Here $\text{MinResLoc}(\varphi)$ is contained in $\{z \in \mathbb{H}_{\text{Berk}} : \rho(\zeta_G, z) \leq 1\}$, while $\frac{2}{d-1} \text{ordRes}(\varphi) = 6$. For $0 \leq \alpha \leq 1$, we have $\varphi(\zeta_{D(0,p^\alpha)}) = \zeta_{D(0,p^{2\alpha+1})}$, so no point of $\text{MinResLoc}(\varphi)$ is fixed by φ .

3. DISCUSSION, APPLICATIONS, AND QUESTIONS

Examples 2.1, 2.2, and 2.6 show that $\text{MinResLoc}(\varphi)$ need not be contained in the tree spanned by the fixed points alone, or the poles alone. Examples 2.3, 2.5, and 2.7 show that when d is odd, $\text{MinResLoc}(\varphi)$ can be either a point or a segment.

When φ has potential good reduction, $\text{MinResLoc}(\varphi)$ consists of a single point, which is necessarily fixed by φ . When φ does not have potential good reduction, $\text{MinResLoc}(\varphi)$ may or may not contain fixed points. In Example 2.6 it consists of a single point, which is fixed. In Example 2.5, it consists of a segment, which is pointwise fixed. In Examples 2.3 and 2.4, it consists of a single point, which is not fixed; in Example 2.7, it consists of a segment, of which no point is fixed.

In the examples, $\text{MinResLoc}(\varphi)$ lies well inside $\{z \in \mathbb{H}_{\text{Berk}} : \rho(\zeta_G, z) \leq \frac{2}{d-1} \text{ordRes}(\varphi)\}$. Probably the radius $\frac{2}{d-1} \text{ordRes}(\varphi)$ given by Theorem 0.1 is not sharp.

Rationality Considerations. Let H be a subfield of K . Throughout this subsection, we will assume $\varphi(z) \in H(z)$.

We will say that a point $Q \in \mathbb{P}_{\text{Berk}}^1$ is *rational over H* if it is type I point in $\mathbb{P}^1(H)$ or is a type II point corresponding to a disc $D(b, r)$ with $b \in H$ and radius $r \in |H^\times|$. A type II point is rational over H if and only if it belongs to the tree spanned by $\mathbb{P}^1(H)$ and corresponds to a disc with radius $r \in |H^\times|$. The following proposition shows the H -rational type II points are those which can be reached from ζ_G by an element of $\text{GL}_2(H)$; it also shows that the notion of H -rationality for type II points is invariant under H -rational changes of coordinates.

Proposition 3.1. *A type II point Q is rational over H if and only if $Q = \gamma(\zeta_G)$ for some $\gamma \in \text{GL}_2(H)$.*

Proof. If Q is rational over H , it corresponds to a disc $D(b, |a|)$ where $b \in H$ and $a \in H^\times$. Put $\gamma = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in \text{GL}_2(H)$; then $Q = \gamma(\zeta_G)$. Conversely, suppose $Q = \gamma(\zeta_G)$ where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(H)$. Write \mathcal{O}_H for the ring of integers of H . Multiplying γ on the right by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \text{GL}_2(\mathcal{O}_H)$ interchanges the columns of γ , so without loss we can assume $|c| \leq |d|$. Then, multiplying γ on the right by $\begin{bmatrix} 1 & 0 \\ -c/d & 1 \end{bmatrix} \in \text{GL}_2(\mathcal{O}_H)$ brings it to the form $\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} \in \text{GL}_2(H)$. Since $\text{GL}_2(\mathcal{O}_H)$ stabilizes ζ_G , Q corresponds to the disc $D(b_1/d_1, |a_1/d_1|)$, and is rational over H . \square

Let $\text{Aut}^c(K/H)$ be the group of continuous automorphisms of K fixing H . The natural action of $\text{Aut}^c(K/H)$ on $\mathbb{P}^1(K)$ extends to an action on $\mathbb{P}_{\text{Berk}}^1$ which preserves the type of each point. On points of type II or III, the action can be described as follows: if $\sigma \in \text{Aut}^c(K/H)$ and Q corresponds to the disc $D(b, r)$, then $\sigma(Q)$ corresponds to $D(\sigma(b), r)$. The image disc is well-defined, since for any $b' \in K$ with $D(b', r) = D(b, r)$ we have $|\sigma(b') - \sigma(b)| = |b' - b| \leq r$. For a point Q of type IV, if Q corresponds to a sequence of nested discs $\{D(a_i, r_i)\}_{i \geq 0}$ under Berkovich's classification theorem, then $\sigma(Q)$ corresponds to the sequence of nested discs $\{D(\sigma(a_i), r_i)\}_{i \geq 0}$.

If a type II point is rational over H , clearly it is fixed by each $\sigma \in \text{Aut}^c(K/H)$. However, the converse is not true: a type II point fixed by $\text{Aut}^c(K/H)$ need not be H -rational. Indeed, each point in the tree spanned by $\mathbb{P}^1(H)$, with radius in $|H^\times|$ or not, is fixed by $\text{Aut}^c(K/H)$. There can also be type II points in $\mathbb{P}_{\text{Berk}}^1$ outside the tree spanned by $\mathbb{P}^1(H)$ which are fixed by $\text{Aut}^c(K/H)$. For example, if $H = \mathbb{Q}_2$ and $K = \mathbb{C}_2$, then $\zeta_{D(i, 1/2)}$ is fixed by $\text{Aut}^c(\mathbb{C}_2/\mathbb{Q}_2)$ since $|\sigma(i) - i| \leq 1/2$ for each $\sigma \in \text{Aut}^c(\mathbb{C}_2/\mathbb{Q}_2)$. However $D(i, 1/2) \cap \mathbb{Q}_2$ is empty: $|x - i| \geq 1/\sqrt{2}$ for each $x \in \mathbb{Q}_2$. Thus $\zeta_{D(i, 1/2)}$ is not in the tree spanned by $\mathbb{P}^1(\mathbb{Q}_2)$.

The action of $\sigma \in \text{Aut}^c(K/H)$ on $\mathbb{P}_{\text{Berk}}^1$ is continuous for the strong topology: indeed, the description of the action shows that for all $x, y \in \mathbb{H}_{\text{Berk}}$, one has $\rho(\sigma(x), \sigma(y)) = \rho(x, y)$. It follows that σ takes paths to paths: if $[x, y]$ is a path with endpoints in \mathbb{H}_{Berk} , then for each $Q \in \mathbb{H}_{\text{Berk}}^1$ we have $Q \in [x, y]$ iff $\rho(x, y) = \rho(x, Q) + \rho(Q, y)$; thus $Q \in [x, y]$ iff $\sigma(Q) \in [\sigma(x), \sigma(y)]$. If $[x, y]$ has one or both endpoints in $\mathbb{P}^1(K)$, it can be exhausted by an increasing sequence of paths with endpoints in \mathbb{H}_{Berk} , so we still have $\sigma([x, y]) = [\sigma(x), \sigma(y)]$.

Proposition 3.2. *For all $\varphi(z) \in K(z)$, all $\sigma \in \text{Aut}^c(K/H)$, and all $Q \in \mathbb{P}_{\text{Berk}}^1$, we have $\sigma(\varphi(Q)) = (\sigma(\varphi))(\sigma(Q))$.*

Proof. Given $\varphi(z) \in K(z)$ and $\sigma \in \text{Aut}^c(K/H)$, if Q is of type I the assertion is clear. If Q is of type II and corresponds to a disc $D(b, r)$, the assertion follows from the case of type I points and the description of the action of φ on generic type I points in $D(b, r)$ given in ([2], Proposition 2.18). Finally, if Q is of type III or IV, the assertion follows from the case of type II points and continuity. \square

In particular, $\sigma(\gamma(Q)) = \gamma(\sigma(Q))$ for all $\gamma \in \text{GL}_2(H)$ and $\sigma \in \text{Aut}^c(K/H)$. This shows the action of $\text{Aut}^c(K/H)$ on $\mathbb{P}_{\text{Berk}}^1$ is independent of H -rational changes of coordinates.

We will say that a subset $X \subset \mathbb{P}_{\text{Berk}}^1$ is *stable under* $\text{Aut}^c(K/H)$ if $\sigma(x) \in X$ for each $x \in X$ and $\sigma \in \text{Aut}^c(K/H)$, that X is *pointwise fixed by* $\text{Aut}^c(K/H)$ if $\sigma(x) = x$ for each $x \in X$ and $\sigma \in \text{Aut}^c(K/H)$.

Proposition 3.3. *If φ is rational over a subfield $H \subset K$, then $\text{MinResLoc}(\varphi)$ is stable under $\text{Aut}^c(K/H)$, and it contains at least one point fixed by $\text{Aut}^c(K/H)$. However, $\text{MinResLoc}(\varphi)$ need not contain points of the tree spanned by $\mathbb{P}^1(H)$, and it need not be pointwise fixed by $\text{Aut}^c(K/H)$. On the other hand, if $\deg(\varphi)$ is odd, $\text{MinResLoc}(\varphi)$ can contain arbitrarily many H -rational type II points.*

Proof. If φ is rational over H , then $\text{ordRes}_\varphi(\sigma(Q)) = \text{ordRes}_\varphi(Q)$ for all $\sigma \in \text{Aut}^c(K/H)$ and all $Q \in \mathbb{P}_{\text{Berk}}^1$. Thus, $\text{MinResLoc}(\varphi)$ is stable under $\text{Aut}^c(K/H)$. To see that $\text{MinResLoc}(\varphi)$ always contains at least one point fixed by $\text{Aut}^c(K/H)$, note that if $\text{MinResLoc}(\varphi)$ consists of a single point, $\text{Aut}^c(K/H)$ fixes that point. On the other hand, if $\text{MinResLoc}(\varphi)$ is a segment, then since $\text{Aut}^c(K/H)$ preserves path distances, each $\sigma \in \text{Aut}^c(K/H)$ must either leave $\text{MinResLoc}(\varphi)$ pointwise fixed, or flip it end-to-end; in either case σ fixes the midpoint of $\text{MinResLoc}(\varphi)$.

Example 2.2, with $\varphi(z) = (z^2 - z)/(2z)$ and $H = \mathbb{Q}_2$, shows that $\text{MinResLoc}(\varphi)$ can be pointwise fixed by $\text{Aut}^c(K/H)$ without meeting the tree spanned by $\mathbb{P}^1(H)$: $\text{MinResLoc}(\varphi) = \{\zeta_{D(i, 1/2)}\}$, and $\zeta_{D(i, 1/2)}$ does not belong to the tree spanned by $\mathbb{P}^1(H)$, as shown above. It would be interesting to know how far off the tree $\text{MinResLoc}(\varphi)$ can lie.

Example 2.5, with $\varphi(z) = \frac{p^n z^3 + z^2 - p^n z}{-p^n z^2 + z + p^n}$ and $H = \mathbb{Q}_p$, shows that when $d = \deg(\varphi)$ is odd, $\text{MinResLoc}(\varphi)$ can contain arbitrarily many type II points rational over H : in that example $\text{MinResLoc}(\varphi)$ is a segment of path-length $2n$ contained in the path $[0, \infty]$ with H -rational endpoints.

It is also possible for $\text{MinResLoc}(\varphi)$ to be a segment “orthogonal to” the tree spanned by $\mathbb{P}^1(H)$: take $H = \mathbb{Q}_p$ with p odd. If $a \in \mathbb{Z}_p^\times$ is a non-square unit, and $u = \sqrt{a}$, then the function $\varphi_1(z) = \frac{-z^3 + (4p^n + 1)u^2 z}{(4p^n - 1)z^2 + u^2}$ from Example 2.5 is \mathbb{Q}_p -rational. Its minimal resultant locus is $[\zeta_{D(-u, 1/p^n)}, \zeta_{D(u, 1/p^n)}]$, which meets the tree spanned by $\mathbb{P}^1(\mathbb{Q}_p)$ only at the \mathbb{Q}_p -rational type II point ζ_G . Likewise, the function $\varphi_2(z) = \frac{-z^3 + (4p^n + 1)u^2 p z}{(4p^n - 1)z^2 + p u^2}$ from Example 2.5 is \mathbb{Q}_p -rational. Its minimal resultant locus meets the tree spanned by $\mathbb{P}^1(\mathbb{Q}_p)$ at $\zeta_{D(0, p^{-1/2})}$, but that point is not \mathbb{Q}_p -rational because its radius does not belong to $|\mathbb{Q}_2^\times|$. In both examples, each $\sigma \in \text{Aut}^c(K/H)$ with $\sigma(\sqrt{a}) = -\sqrt{a}$ flips $\text{MinResLoc}(\varphi)$ end-to-end; the midpoint of $\text{MinResLoc}(\varphi)$ is the only point fixed by $\text{Aut}^c(K/H)$. \square

Now assume that H is discretely valued: in this case, the H -rational type II points are discrete in \mathbb{H}_{Berk} for the strong topology, and the subtree of $\mathbb{P}_{\text{Berk}}^1$ spanned by $\mathbb{P}^1(H)$ is branched at precisely the H -rational type II points.

If Q is a type II point rational over H , the action of $\text{Aut}^c(K/H)$ on $\mathbb{P}_{\text{Berk}}^1$ induces an action of $\text{Aut}^c(K/H)$ on the tangent space T_Q , which takes the class of a path $[Q, x]$ to the class of $[Q, \sigma(x)]$. This is well-defined, since if x and x' belong to the same tangent direction at Q , then the paths $[Q, x]$ and $[Q, x']$ share an initial segment; thus $[Q, \sigma(x)]$ and $[Q, \sigma(x')]$ share an initial segment as well.

The following proposition shows that if φ is rational over H , and if $Q \notin \text{MinResLoc}(\varphi)$ is a type II point rational over H , then $\text{MinResLoc}(\varphi)$ lies in a tangent direction at Q fixed by $\text{Aut}^c(K/H)$. When $H = H_v$ is a local field, we will use this in giving a steepest descent algorithm for finding an H_v -rational point where $\text{ordRes}_\varphi(\cdot)$ is minimal for H_v -rational points.

Proposition 3.4. *Suppose H_v is a local field and φ is rational over H_v . Let Q be an H_v -rational type II point not contained in $\text{MinResLoc}(\varphi)$. Then $\text{MinResLoc}(\varphi)$ lies in a tangent direction at Q coming from the tree spanned by $\mathbb{P}^1(H_v)$.*

Proof. If H_v has residue field \mathbb{F}_q , then T_Q is parametrized by $\mathbb{P}^1(\overline{\mathbb{F}_q})$ and the tangent directions at Q fixed by $\text{Aut}^c(K/H_v)$ correspond to the points of $\mathbb{P}^1(\mathbb{F}_q)$. These are precisely the tangent directions at Q coming from the tree spanned by $\mathbb{P}^1(H_v)$. (We remark that even if H is not a local field, the conclusion of the proposition will hold if the residue field of K is separable over the residue field of H .) \square

If H_v is a local field and $\text{MinResLoc}(\varphi)$ contains no H_v -rational type II points, there are exactly two H_v -rational type II points adjacent to it in the tree spanned by $\mathbb{P}^1(H_v)$. The function $\text{ordRes}_\varphi(\cdot)$ may take the same or different values at those points; its value is strictly larger at all other H_v -rational type II points. Example 2.2 gives a case where the minimum is taken on at one of the two adjacent H -rational type II points, and Example 2.4 gives a case where it is taken on at both points.

Bounds for the degree of an extension where φ^γ has Minimal Resultant.

It is useful to note that in Theorem 0.1, the tree $\Gamma_{\text{Fix}, \varphi^{-1}(\infty)}$ can be replaced by the tree $\Gamma_{\text{Fix}, \varphi^{-1}(a)}$ spanned by the fixed points of φ and the preimages of a , for any $a \in \mathbb{P}^1(K)$:

Proposition 3.5. *For each $a \in \mathbb{P}^1(K)$, $\text{MinResLoc}(\varphi)$ is contained in the tree $\Gamma_{\text{Fix}, \varphi^{-1}(a)}$ spanned by the fixed points of φ and the set of preimages $\{z \in \mathbb{P}^1(K) : \varphi(z) = a\}$.*

Proof. Take $a \in \mathbb{P}^1(K)$, and choose $\gamma \in \text{GL}_2(K)$ with $\gamma(\infty) = a$. It follows from (22) that

$$\text{MinResLoc}(\varphi) = \gamma(\text{MinResLoc}(\varphi^\gamma)) .$$

By Theorem 0.1, $\text{MinResLoc}(\varphi^\gamma)$ is contained in the tree $\Gamma_{\text{Fix}, (\varphi^\gamma)^{-1}(\infty)}$ spanned by the fixed points and poles of φ^γ , so $\text{MinResLoc}(\varphi)$ is contained in the tree $\gamma(\Gamma_{\text{Fix}, (\varphi^\gamma)^{-1}(\infty)})$. By equivariance, Q is a fixed point of φ^γ if and only if $\gamma(Q)$ is a fixed point of φ , and P is a pole of φ^γ if and only if $\varphi(\gamma(P)) = a$. Thus $\gamma(\Gamma_{\text{Fix}, (\varphi^\gamma)^{-1}(\infty)}) = \Gamma_{\text{Fix}, \varphi^{-1}(a)}$. \square

Let $M = \min_{Q \in \mathbb{P}^1_{\text{Berk}}} (\text{ordRes}_\varphi(Q)) = \min_{\gamma \in \text{GL}_2(K)} (\text{ordRes}(\varphi^\gamma))$.

Theorem 3.6. *Let H be a subfield of K , and suppose $\varphi(z) \in H(z)$ has degree $d \geq 2$. Then there is an extension L/H in K with $[L : H] \leq (d+1)^2$ such that $\text{ordRes}(\varphi^\gamma) = M$ for some $\gamma \in \text{GL}_2(L)$.*

Proof. It is enough to show there is an extension L/H with $[L : H] \leq (d+1)^2$ such that $\text{MinResLoc}(\varphi)$ contains a type II point Q rational over L .

Put $a = \varphi(\infty) \in \mathbb{P}^1(H)$. Let F_1, \dots, F_{d+1} be the fixed points of φ , and let A_1, \dots, A_d be the preimages of a under φ , listed with multiplicity. Without loss we can assume that $A_1 = \infty$. By Proposition 3.5, $\text{MinResLoc}(\varphi)$ is contained in the tree $\Gamma_{\text{Fix}, \varphi^{-1}(a)}$, which is the union of the paths $[F_i, \infty]$ and $[A_j, \infty]$ for $i = 1, \dots, d+1$, $j = 2, \dots, d$. Let Q be an endpoint of $\text{MinResLoc}(\varphi)$, and let $P \in \{F_1, \dots, F_{d+1}, A_2, \dots, A_d\}$ be such that $Q \in [P, \infty]$. Put $L_0 = H(P)$. We have $[H(F_i) : H] \leq d+1$ for each i , and $[H(A_j) : H] \leq d-1$ for each j , so $[L_0 : H] \leq d+1$. Fix $\gamma \in \text{GL}_2(L_0)$ with $\gamma(0) = P$ and $\gamma(\infty) = \infty$, and let $Q_0 = \gamma^{-1}(Q) \in [0, \infty]$. By the discussion after formula (14), there are an $\alpha \in L_0^\times$ and an integer e with $1 \leq e \leq d+1$ such that $Q_0 = \zeta_{D(0, |\alpha|^{1/e})}$. Put $L = L_0(\alpha^{1/e})$. Then Q is rational over L , and $[L : H] \leq (d+1)^2$. \square

Corollary 3.7. *For each $d \geq 2$, there is a first order formula $\mathcal{F}_d(f_0, \dots, f_d, g_0, \dots, g_d)$ in the language of valued fields such that if H is a Henselian nonarchimedean valued field, and if $\varphi(z) = (f_d z^d + \dots + f_0)/(g_d z^d + \dots + g_0) \in H(z)$, then φ has potential good reduction if and only if $H \models \mathcal{F}_d(f_0, \dots, f_d, g_0, \dots, g_d)$.*

Proof. If H is Henselian (in particular, if H is complete), then for each finite extension $H(\beta)/H$ there is a unique extension of the valuation $\text{ord}(\cdot)$ on H to a valuation on $H(\beta)$, given by $\text{ord}_{H(\beta)}(z) = (1/m)\text{ord}(N_{H(\beta)/H}(z))$ for $z \in H(\beta)$, where $[H(\beta) : H] = m$. If $z = a_0 + a_1\beta + \dots + a_{m-1}\beta^{m-1}$ with $a_0, \dots, a_{m-1} \in H$, then $N_{H(\beta)/H}(z)$ is a universal polynomial in the a_i and the coefficients of the minimal polynomial of β over H .

Write (F, G) for the natural representation of φ . Let $\mathcal{F}_{d,0}(f_0, \dots, f_d, g_0, \dots, g_d)$ be the formula “ $\text{Res}(F, G) \neq 0$ ”, and for $m = 1, \dots, (d+1)^2$ let $\mathcal{F}_{d,m}(f_0, \dots, f_d, g_0, \dots, g_d)$ be the formula

“ There exist $a_1, \dots, a_m \in H$ such that $h_m(x) = x^m + a_1 x^{m-1} + \dots + a_m$ is irreducible over H , and there exist a root β of $h_m(x)$

and $a, b, c, d \in H(\beta)$ with $ad - bc \neq 0$, such that for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $\text{ord}_{H(\beta)}(\text{Res}(F^\gamma, G^\gamma)) - 2d \min(\text{ord}_{H(\beta)}(F^\gamma), \text{ord}_{H(\beta)}(G^\gamma)) = 0$."

We can take \mathcal{F}_d to be $\mathcal{F}_{d,0} \wedge (\mathcal{F}_{d,1} \vee \cdots \vee \mathcal{F}_{d,(d+1)^2})$. \square

Failure to achieve the Minimal Resultant over the Field of Moduli.

Suppose $\varphi(z) \in H(z)$, where $H \subset K$. Let $\mathcal{F}_H(\varphi)$ be the set of fields L with $H \subseteq L \subseteq K$ for which there is some $\gamma \in \text{GL}_2(L)$ such that $\text{ordRes}(\varphi^\gamma)$ is minimal. When $\text{MinResLoc}(\varphi) = \{Q\}$ consists of a single point, $\mathcal{F}_H(\varphi)$ is the set of fields $H \subseteq L \subseteq K$ such that there is some $\gamma \in \text{GL}_2(L)$ with $\gamma(\zeta_G) = Q$. The *field of moduli* for the minimal resultant problem is

$$H_\varphi := \bigcap_{L \in \mathcal{F}_H(\varphi)} L.$$

It is natural to ask if there is a $\gamma \in \text{GL}_2(H_\varphi)$ for which $\text{ordRes}(\varphi^\gamma)$ is minimal. If $\text{MinResLoc}(\varphi)$ contains an H -rational point, the answer is trivially yes. If $\text{MinResLoc}(\varphi)$ contains no H -rational points, the answer is generally no. In Example 2.1, take $d = p > 2$, with $\varphi(z) = \frac{z^p - p}{z^{p-1}}$ and $H = \mathbb{Q}_p$. We have $\text{MinResLoc}(\varphi) = \{Q\}$ where $Q = \zeta_{D(0, p^{1/p})}$. Here Q is rational over L if and only if the value group of L contains $p^{1/p}$. In particular, Q is rational over $L_1 = \mathbb{Q}_p(\sqrt[p]{p})$ and over $L_2 = \mathbb{Q}_p(\zeta_p \sqrt[p]{p})$ where ζ_p is any primitive p^{th} root of unity. Since $p > 2$, necessarily $L_1 \cap L_2 = \mathbb{Q}_p$ (otherwise $L_1 = L_2$, since both extensions have degree p ; but then $\zeta_p \in L_1$, so $p - 1 = [\mathbb{Q}_p(\zeta_p) : \mathbb{Q}_p]$ divides $[L_1 : \mathbb{Q}_p] = p$). Thus $H_\varphi = \mathbb{Q}_p$. However, $p^{1/p}$ is not in the value group of \mathbb{Q}_p^\times , so by Proposition 3.1 there can be no $\gamma \in \text{GL}_2(\mathbb{Q}_p)$ with $\gamma(\zeta_G) = Q$. Likewise, in Example 2.2, for $\varphi(z) = \frac{z^2 - 1}{2z}$ and $H = \mathbb{Q}_2$, we have $\text{MinResLoc}(\varphi) = \{Q\}$ where $Q = \zeta_{D(i, 1/2)}$. Here $D(i, 1/2) = D(\sqrt{3}, 1/2)$ since $|i - \sqrt{3}| = 1/2$, so $Q = \gamma_1(\zeta_G) = \gamma_2(\zeta_G)$ where $\gamma_1 = \begin{bmatrix} 2 & i \\ 0 & 1 \end{bmatrix}$ and $\gamma_2 = \begin{bmatrix} 2 & \sqrt{3} \\ 0 & 1 \end{bmatrix}$. Since $\mathbb{Q}_2(i) \cap \mathbb{Q}_2(\sqrt{3}) = \mathbb{Q}_2$, we have $H_\varphi = \mathbb{Q}_2$. However, $D(i, 1/2) \cap \mathbb{Q}_2$ is empty. Hence there can be no $\gamma \in \text{GL}_2(\mathbb{Q}_2)$ with $\gamma(\zeta_G) = Q$.

Answers to questions of Silverman concerning global Minimal Models.

Throughout this subsection, H will be a number field, and $\varphi(z) \in H(z)$ will have degree $d \geq 2$. Let \mathcal{O}_H be the ring of integers of H . Given a nonarchimedean place v of H , let H_v be the completion of H at v , \mathcal{O}_v the valuation ring of H_v , and π_v a generator for the maximal ideal of \mathcal{O}_v . Let \mathbb{C}_v be the completion of the algebraic closure of H_v . We will write $\text{ord}_v(\cdot)$ for the valuation on \mathbb{C}_v normalized so that $\text{ord}_v(\pi_v) = 1$, and $\text{ordRes}_v(\varphi)$ and $\text{ordRes}_{\varphi,v}(\cdot)$ for the functions previously denoted $\text{ordRes}(\varphi)$ and $\text{ordRes}_\varphi(\cdot)$. In this way the theory developed above is applicable for each nonarchimedean place v of H .

A representation (F, G) of φ with $F(X, Y), G(X, Y) \in H[X, Y]$ is called a *representation of φ over H* ; such a pair is unique up to scaling by an element of H^\times . One can always arrange that $F, G \in \mathcal{O}_H[X, Y]$; in that case, the representation is called *integral*.

In ([25], §4.11), Silverman asks if (and when) it is possible to choose an "optimal" integral representation for φ , analogous to a minimal Weierstrass model for an elliptic curve. For each prime $\mathfrak{p} = \mathfrak{p}_v$ of \mathcal{O}_H , he defines an integer

$$\varepsilon_{\mathfrak{p}}(\varphi) = \min_{\gamma \in \text{GL}_2(H)} \text{ordRes}_v(\varphi^\gamma) \geq 0.$$

He then defines “global minimal resultant” of φ to be the ideal

$$\mathfrak{R}_\varphi = \prod_{\mathfrak{p}} \mathfrak{p}^{\varepsilon_{\mathfrak{p}}(\varphi)}.$$

Here the product is finite since for a given representation (F, G) of φ over H , for all but finitely many \mathfrak{p} we have $\text{ord}_{\mathfrak{p}}(\text{Res}(F, G)) = 0$.

Given a representation (F, G) for φ over H , in ([25], Proposition 4.99) Silverman shows there is a fractional ideal $\mathfrak{a}_{F,G}$ of H such that

$$\mathfrak{R}_\varphi = \begin{cases} \mathfrak{a}_{F,G}^{2d} \cdot (\text{Res}(F, G)) & \text{if } d \text{ is odd,} \\ \mathfrak{a}_{F,G}^d \cdot (\text{Res}(F, G)) & \text{if } d \text{ is even.} \end{cases}$$

Let $I(K)$ be the group of fractional ideals of H , and $P(K)$ the group of principal fractional ideals. Silverman shows that if d is odd, the ideal class $[\mathfrak{a}_\varphi] := [\mathfrak{a}_{F,G}] \in I(K)/P(K)$ is independent of the choice of (F, G) , while if d is even, the refined ideal class $[\mathfrak{a}_\varphi] := [\mathfrak{a}_{F,G}] \in I(K)/\{(\alpha)^2 : (\alpha) \in P(K)\}$ is independent of the choice of (F, G) . He calls $[\mathfrak{a}_\varphi]$ the *Weierstrass class of φ over H* .

We will say that φ has a *global minimal model over H* if for some $\gamma \in \text{GL}_2(H)$, the function φ^γ has an integral representation (F^γ, G^γ) over H such that

$$\text{ord}_{\mathfrak{p}}(\text{Res}(F^\gamma, G^\gamma)) = \varepsilon_{\mathfrak{p}}(\varphi) \quad \text{for each prime } \mathfrak{p} \text{ of } \mathcal{O}_H.$$

In ([25], Proposition 4.100), Silverman shows that if φ has a global minimal model over H , then the Weierstrass class $\bar{\mathfrak{a}}_\varphi$ is trivial. In ([25], Exercise 4.46) he asks

- (a) When $H = \mathbb{Q}$, does every $\varphi(z) \in \mathbb{Q}(z)$ of degree $d \geq 2$ have a global minimal model over \mathbb{Q} ?
- (b) When H is an arbitrary number field and $\varphi(z) \in H(z)$ has degree $d \geq 2$, if S is a finite set of primes of \mathcal{O}_H such that the localization $\mathcal{O}_{H,S}$ is a Principal Ideal Domain, does φ have a *global S -minimal model*? In other words, is there a $\gamma \in \text{GL}_2(H)$ such that φ^γ has a representation (F^γ, G^γ) with $F^\gamma(X, Y), G^\gamma(X, Y) \in \mathcal{O}_{H,S}[X, Y]$, satisfying

$$\text{ord}_{\mathfrak{p}}(\text{Res}(F^\gamma, G^\gamma)) = \varepsilon_{\mathfrak{p}}(\varphi) \quad \text{for each prime } \mathfrak{p} \notin S?$$

- (c) When H is an arbitrary number field and $\varphi(z) \in H(z)$ has degree $d \geq 2$, if the Weierstrass class $[\mathfrak{a}_\varphi]$ is trivial, does φ have a global minimal model over H ?

As has already been noted by Bruin and Molnar ([7]), the answer to the first two questions is “Yes”. This follows from the Strong Approximation Theorem and the fact that the subgroup $\text{Aff}_2(K) \subset \text{GL}_2(K)$ acts transitively on the type II points in $\mathbb{P}_{\text{Berk}}^1$. Indeed, in (b), let $\tilde{S} \supseteq S$ be a finite set of primes such that φ has good reduction outside \tilde{S} . For each prime $\mathfrak{p} = \mathfrak{p}_v \in \tilde{S}$, choose a $\gamma_{\mathfrak{p}} \in \text{GL}_2(H)$ such that $\text{ord}_{\mathfrak{p}}(\varphi^{\gamma_{\mathfrak{p}}}) = \varepsilon_{\mathfrak{p}}$ and put $\xi_{\mathfrak{p}} = \gamma_{\mathfrak{p}}(\zeta_G)$. By Proposition 3.1, $\xi_{\mathfrak{p}} \in \mathbb{P}_{\text{Berk},v}^1$ is rational over H ; thus there exist $a_{\mathfrak{p}}, b_{\mathfrak{p}} \in H$ with $a_{\mathfrak{p}} \neq 0$, such that $\xi_{\mathfrak{p}} = \zeta_{D(b_{\mathfrak{p}}, |a_{\mathfrak{p}}|_v)}$. Since $\mathcal{O}_{H,S}$ is a PID there is an $a \in H$ such that $\text{ord}_{\mathfrak{p}}((a)) = \text{ord}_{\mathfrak{p}}((a_{\mathfrak{p}}))$ for each $\mathfrak{p} \in \tilde{S}$ and $\text{ord}_{\mathfrak{p}}((a)) = 0$ for each $\mathfrak{p} \notin \tilde{S}$. By the Strong Approximation Theorem there is a $b \in H$ such that $\text{ord}_{\mathfrak{p}}(b - b_{\mathfrak{p}}) > \text{ord}(a_{\mathfrak{p}})$ for each $\mathfrak{p} \in \tilde{S}$ and $\text{ord}_{\mathfrak{p}}(b) = 0$ for each $\mathfrak{p} \notin \tilde{S}$. Put $\gamma = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$; then $\gamma(\zeta_G) = \xi_{\mathfrak{p}}$ for each $\mathfrak{p} \in \tilde{S}$ and $\gamma(\zeta_G) = \zeta_G$ for each $\mathfrak{p} \notin \tilde{S}$, so $\text{ord}_{\mathfrak{p}}(\varphi^\gamma) = \varepsilon_{\mathfrak{p}_v}$ for each prime \mathfrak{p}_v . Let (F^γ, G^γ) be a representation of φ^γ over H ; since $\mathcal{O}_{H,S}$ is a PID, we can assume (F^γ, G^γ)

has been scaled so that $\min(\text{ord}_{\mathfrak{p}}(F^\gamma), \text{ord}_{\mathfrak{p}}(G^\gamma)) = 0$ for each $\mathfrak{p} \notin S$. Then F^γ, G^γ are defined over $\mathcal{O}_{H,S}$, and $\text{ord}_{\mathfrak{p}}(\text{Res}(F^\gamma, G^\gamma)) = \varepsilon_{\mathfrak{p}}$ for each $\mathfrak{p} \notin S$, so (F^γ, G^γ) is a global S -minimal model.

The answer to question (c) is “No” in general. The underlying reason for this is a disconnect between the values of $\text{ordRes}_v(\cdot)$ and the points at which they are taken. To obtain counterexamples, consider polynomials of the form $\varphi(z) = z^d + c$ with $d \geq 2$, $c \in H$. For a given prime $\mathfrak{p} = \mathfrak{p}_v$ of \mathcal{O}_H , if $\text{ord}_v(c) \geq 0$ then $\varphi(z)$ has good reduction at \mathfrak{p} . Suppose $\text{ord}_v(c) < 0$. Then $\text{ordRes}_v(\varphi) = -2d \text{ord}_v(c)$. Computing $\text{ordRes}_{\varphi,v}(\cdot)$ on the path $[0, \infty] \subset \mathbb{P}_{\text{Berk},v}^1$, we find that for each $A \in \mathbb{C}_v^\times$

$$\text{ordRes}_{\varphi,v}(\zeta_{D(0,|A|_v)}) = \max((d - d^2)\text{ord}_v(A), -2d \text{ord}_v(c) + (d + d^2)\text{ord}_v(A)).$$

This is minimal when $\text{ord}_v(A) = (1/d)\text{ord}_v(c)$. If $(1/d)\text{ord}_v(c)$ is not an integer, by convexity the least value of $\text{ordRes}_{\varphi,v}(\cdot)$ on H -rational points in $\mathbb{P}_{\text{Berk},v}^1$ occurs when $\text{ord}_v(A)$ is one of the two integers adjacent to $(1/d)\text{ord}_v(c)$.

For a counterexample when d is odd, take $\varphi(z) = z^5 + 1/(1 + 4\sqrt{-5})$, so $d = 5$ and $c = 1/(1 + 4\sqrt{-5})$, with $H = \mathbb{Q}(\sqrt{-5})$. The field H has class number 2. The ideal $\mathfrak{p} = \mathfrak{p}_v = (3, 1 + \sqrt{-5})$ in \mathcal{O}_H is one of the primes containing (3); it is not principal, but $\mathfrak{p}^4 = (1 + 4\sqrt{-5})$, so $\text{ord}_v(c) = -4$.

Clearly $\varphi(z)$ has good reduction at all primes other than \mathfrak{p} . The least value of $\text{ordRes}_{\varphi,v}(\cdot)$ on H -rational points occurs only when $\text{ord}_v(A) = -1$, and one has

$$20 = \text{ordRes}_{\varphi,v}(\zeta_{D(0,|A|_v)}) < \text{ordRes}_v(\varphi) = 40.$$

The integral representation (F, G) with $F(X, Y) = X^5/c + Y^5$, $G(X, Y) = Y^5/c$ satisfies $(\text{Res}(F, G)) = \mathfrak{p}^{40}$, while $\mathfrak{R}_\varphi = \mathfrak{p}^{20}$. Since $\mathfrak{R}_\varphi = \mathfrak{a}_{F,G}^{10} \cdot (\text{Res}(F, G))$, it follows that $\mathfrak{a}_{F,G} = \mathfrak{p}^{-2} = (1/(2 - \sqrt{-5}))$. Thus the class $[\mathfrak{a}_\varphi]$ is trivial. However, there is no $\gamma \in \text{GL}_2(H)$ for which $\text{ordRes}_v(\varphi^\gamma) = \mathfrak{R}_\varphi$. If there were, in $\mathbb{P}_{\text{Berk},v}^1$ we would have $\gamma(\zeta_G) = \zeta_{D(0,3)}$, while for each finite place $w \neq v$, in $\mathbb{P}_{\text{Berk},w}^1$ we would have $\gamma(\zeta_G) = \zeta_G$. By the proof of Proposition 3.1, this would mean that $\text{ord}_v(\det(\gamma)) = -1$ and $\text{ord}_w(\det(\gamma)) = 0$ for all $w \neq v$, so $(\det(\gamma)) = \mathfrak{p}^{-1}$. This is a contradiction since \mathfrak{p}^{-1} is not principal.

For a counterexample when d is even, take $\varphi(z) = z^4 + 1/(19 + 4\sqrt{-23})$, so $d = 4$ and $c = 1/(19 + 4\sqrt{-23})$, with $H = \mathbb{Q}(\sqrt{-23})$. The field H has class number 3. The ideal $\mathfrak{p} = \mathfrak{p}_v = (3, (1 + \sqrt{-23})/2)$ in \mathcal{O}_H is one of the primes containing (3); it is not principal, but $\mathfrak{p}^3 = (2 - \sqrt{-23})$ and $\mathfrak{p}^6 = (19 + 4\sqrt{-23})$, so $\text{ord}_v(c) = -6$.

Clearly $\varphi(z)$ has good reduction at all primes other than \mathfrak{p} . The least value of $\text{ordRes}_{\varphi,v}(\cdot)$ on H -rational points occurs only when $\text{ord}_v(A) = -2$, and one has

$$24 = \text{ordRes}_{\varphi,v}(\zeta_{D(0,|A|_v)}) < \text{ordRes}_v(\varphi) = 48.$$

The normalized representation (F, G) with $F(X, Y) = X^4/c + Y^4$, $G(X, Y) = Y^4/c$ satisfies $(\text{Res}(F, G)) = \mathfrak{p}^{48}$, while $\mathfrak{R}_\varphi = \mathfrak{p}^{24}$. Since $\mathfrak{R}_\varphi = \mathfrak{a}_{F,G}^4 \cdot (\text{Res}(F, G))$, it follows that $\mathfrak{a}_{F,G} = \mathfrak{p}^{-6} = (1/(2 - \sqrt{-23}))^2$. Thus the class $[\mathfrak{a}_\varphi]$ is trivial. However, there is no $\gamma \in \text{GL}_2(H)$ for which $\text{ordRes}_v(\varphi^\gamma) = \mathfrak{R}_\varphi$. If there were, we would have $\text{ord}_v(\det(\gamma)) = -2$ and $\text{ord}_w(\det(\gamma)) = 0$ for all $w \neq v$, so $(\det(\gamma)) = \mathfrak{p}^{-2}$. This is impossible since \mathfrak{p}^{-2} is not principal.

What is the dynamical significance of the Minimal Resultant Locus?

When φ has potential good reduction, the Minimal Resultant Locus consists of the unique repelling fixed point of φ in \mathbb{H}_{Berk} . It is natural to ask about the dynamical significance of the Minimal Resultant Locus when φ does not have potential good reduction.

We do not know the answer to this. The examples in §2 show it does not always consist of fixed points. Rob Benedetto has remarked that another set which arises naturally in arithmetic dynamics, and is either a point or a segment, is the *Barycenter* of φ , defined to be the set of points $Q \in \mathbb{P}_{\text{Berk}}^1$ which minimize the Arakelov-Green's function $g_\varphi(Q, Q)$ (see [2], §10.2), and can be computed as the set of points $Q \in \mathbb{H}_{\text{Berk}}$ such that each component of $\mathbb{P}_{\text{Berk}}^1 \setminus \{Q\}$ has mass at most 1/2 for the invariant measure μ_φ . (The author thanks Benedetto for pointing this out.) In Example 2.3, when $p = 2$ the Barycenter is the segment $[\zeta_{D(0,1/2)}, \zeta_{D(1,1/2)}]$ while the Minimal Resultant Locus is $\{\zeta_G\}$. In Example 2.6, the Barycenter is $\{\zeta_{D(0,p)}\}$ while the Minimal Resultant Locus is $\{\zeta_{D(0,p^{4/3})}\}$. Thus there is no clear relationship between the Minimal Resultant Locus and the Barycenter.

Some other questions about the Minimal Resultant Locus, which may shed light on the general question of its dynamical meaning, are as follows:

- (1) How are the Minimal Resultant Loci of the iterates $\varphi, \varphi^{(2)}, \varphi^{(3)}, \dots$ related? There are examples where the Minimal Resultant Loci of all the iterates are the same. Does this happen in general? If not, do they stabilize for large n , or converge to something with geometric significance? Where do they lie relative to the Berkovich Julia set of φ ?
- (2) Can one give a geometric description of the Minimal Resultant Locus? This appears necessary in order to address stability questions of the kind above.

By Proposition 3.5, $\text{MinResLoc}(\varphi)$ is contained in the intersection of the trees $\Gamma_{\text{Fix}, \varphi^{-1}(a)}$ for all $a \in \mathbb{P}^1(K)$. Recall that a *repelling fixed point* of φ in \mathbb{H}_{Berk} is a point $x \in \mathbb{H}_{\text{Berk}}$ such that $\varphi(x) = x$ and the degree of the reduction of φ at x is at least 2 (see [2], p.340). In [24] the author shows

Theorem 3.8. *The intersection of the trees $\Gamma_{\text{Fix}, \varphi^{-1}(a)}$ for all $a \in \mathbb{P}^1(K)$ is the tree $\Gamma_{\text{Fix}, \text{Repel}}$ spanned by the fixed points of φ in $\mathbb{P}^1(K)$ and the repelling fixed points of φ in \mathbb{H}_{Berk} .*

Where does the Minimal Resultant Locus lie in this tree? Does it consist of points subject to some balance condition, like the one describing the Barycenter? Is it possible to prune the tree still further? It seems plausible that the Minimal Resultant Locus might belong to the subtree spanned by the attracting and repelling fixed points of φ .

- (3) What is the arithmetic significance of the value of $\text{ordRes}_\varphi(\cdot)$ on $\text{MinResLoc}(\varphi)$? It is clearly a conjugacy invariant which measures the complexity of φ .

4. ALGORITHMS

In this section we give two algorithms: one which computes the Minimal Resultant Locus of φ , and another which finds the H -rational points where $\text{ordRes}_\varphi(\cdot)$ is minimal, in the case when H is a local field and φ is rational over H .

Given $\varphi(z) \in K(z)$ with $d = \deg(\varphi) \geq 2$, put $a = \varphi(\infty) \in K \cup \{\infty\}$. The following algorithm finds the minimal value of $\text{ordRes}_\varphi(\cdot)$ and determines $\text{MinResLoc}(\varphi)$ by working in the tree $\Gamma_{\text{Fix}, \varphi^{-1}(a)}$. This tree is well suited for computations, because it is spanned by ∞ and the finite fixed points and finite solutions to $\varphi(z) = a$. This means the necessary changes of coordinates can be done with conjugacies by affine translations.

**Algorithm A: Minimize $\text{ordRes}_\varphi(\cdot)$, find $\text{MinResLoc}(\varphi)$,
and find a $\gamma \in \text{GL}_2(K)$ for which $\text{ordRes}(\varphi^\gamma)$ is minimal.**

Given a complete nonarchimedean valued field K with absolute value $|x| = q^{-\text{ord}(x)}$, and a function $\varphi(z) \in K(z)$ with $d = \deg(\varphi) \geq 2$:

- (1) [Find the endpoints of $\Gamma_{\text{Fix}, \varphi^{-1}(a)}$.]
 - (a) Write $\varphi(z) = f(z)/g(z)$ with $f(z), g(z) \in K[z]$ and put $a = \varphi(\infty)$.
 - (b) Find the roots of $f(z) - zg(z) = 0$ (the finite fixed points).
 - (c) If $a = \infty$, find the roots of $g(z) = 0$ (the finite poles). If $a \neq \infty$, find the roots of $f(z) - a \cdot g(z) = 0$ (the finite solutions to $\varphi(z) = a$).
 - (d) List the distinct roots from (b) and (c) as $\{\alpha_1, \dots, \alpha_k\}$.
- (2) [Minimize $\text{ordRes}_\varphi(\cdot)$ on each path $[\alpha_i, \infty]$.]

For each $i = 1, \dots, k$, do the following:

 - (a) Put $\gamma_i(z) = z + \alpha_i$.
 - (b) Find a normalized representation (F_i, G_i) for $\varphi^{\gamma_i}(z) = \varphi(z + \alpha_i) - \alpha_i$.
 - (c) Compute $R_i = \text{ord}(\text{Res}(F_i, G_i))$.
 - (d) Writing $F_i(X, Y) = a_d X^d + \dots + a_0 Y^d$, $G_i(X, Y) = b_d X^d + \dots + b_0 Y^d$, put $C_\ell = R_i - 2d \text{ord}(a_\ell)$, $D_\ell = R_i - 2d \text{ord}(b_\ell)$ for $\ell = 0, \dots, d$.
 - (e) Minimize the piecewise affine function

$$\chi_i(t) = \max \left(\max_{0 \leq \ell \leq d} (C_\ell + (d^2 + d - 2d\ell)t), \max_{0 \leq \ell \leq d} (D_\ell + (d^2 + d - 2d(\ell + 1)t)) \right).$$
 - (f) Record the minimum value of $\chi_i(t)$ as M_i , and record the set of points where it is achieved as a singleton $\{\zeta_{D(\alpha_i, r_i)}\}$ or a segment $[\zeta_{D(\alpha_i, r_{i,1})}, \zeta_{D(\alpha_i, r_{i,2})}]$, where $r = q^{-t}$ for a given t .
- (3) [Find the Minimum.] Let $M = \min_{1 \leq i \leq k} M_i$, output “ $\min(\text{ordRes}_\varphi(\cdot)) = M$ ”.
- (4) [Find the Minimal Resultant Locus.] Consider the indices i with $M = M_i$:
 - (a) If for each such i , $\chi_i(t)$ achieved M at a single point, output “ $\text{MinResLoc}(\varphi) = \{\zeta_{D(\alpha_i, r_i)}\}$ ” for any such i , and go to (5).
 - (b) If for some such i , $\chi_i(t)$ achieved M on a segment,
 - (i) Find the relevant nodes of the tree $\Gamma_{\text{Fix}, \varphi^{-1}(a)}$:
for all (i, j) with $1 \leq i < j \leq k$ such that $M = M_i = M_j$,
find $r_{ij} = |\alpha_i - \alpha_j|$, then record $\zeta_{D(\alpha_i, r_{ij})} = \zeta_{D(\alpha_j, r_{ij})}$ as a node.
 - (ii) Using the nodes, collate the segments $[\zeta_{D(\alpha_i, r_{i,1})}, \zeta_{D(\alpha_i, r_{i,2})}]$
into a single segment $[\zeta_{D(a, r_a)}, \zeta_{D(b, r_b)}]$,
and output “ $\text{MinResLoc}(\varphi) = [\zeta_{D(a, r_a)}, \zeta_{D(b, r_b)}]$ ”.
- (5) [Find $\gamma \in \text{GL}_2(K)$ with $\text{ordRes}(\varphi^\gamma) = M$.]
 - (a) Choose an endpoint of $\text{MinResLoc}(\varphi)$
and write it as $\zeta_{D(B, |A|)}$ with $A \in K^\times$, $B \in K$.
 - (b) Output “ $\gamma = \begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix}$ ”, then halt.

The correctness of Algorithm A follows from Theorem 0.1 and Proposition 3.5.

When $\varphi(z) \in \mathbb{Q}(z)$ and $|x| = |x|_p$ for a rational prime p , Algorithm A could be implemented either using arithmetic over global fields or over local fields. Working over global fields, one has the advantage of exact results, but care is needed to avoid coefficient explosion in intermediate steps. Over local fields, the implementation is more transparent and coefficient explosion does not occur, but careful error estimates are needed to assure that the results are correct. Below we sketch a possible implementation using arithmetic in global fields. An implementation using local fields could be given using Theorem 0.2 and the factoring algorithm of Cantor and Gordon ([8]), which runs in probabilistic polynomial time and provides explicit error estimates for the precision needed. See also the factoring algorithms of Pauli ([21], [22]) and the references therein.

Take $K = \mathbb{C}_p$, and normalize the valuation $\text{ord}(\cdot)$ on \mathbb{C}_p so it extends the valuation $\text{ord}_p(\cdot)$ on \mathbb{Q} . Let $\alpha_1, \dots, \alpha_k$ be the roots from Step (1), and put $L = \mathbb{Q}(\alpha_1, \dots, \alpha_k)$. Up to the action of $\text{Aut}^c(\mathbb{C}_p/\mathbb{Q}_p)$, embeddings of L in \mathbb{C}_p correspond to primes of \mathcal{O}_L above p . Since L/\mathbb{Q} is galois, it suffices to find one of those primes \mathfrak{p} , and work with the corresponding valuation $\text{ord}_{\mathfrak{p}}(\cdot)$ on L . However, implementing Algorithm A does not require computing the full ring of integers \mathcal{O}_L : it is enough to find a p -maximal order $\mathcal{O}_{L,p} \subset \mathcal{O}_L$ and a maximal ideal of that order lying over (p) . It is beneficial to localize at p , and work over $\mathbb{Z}_{(p)}$ rather than \mathbb{Z} : the localization $\mathcal{O}_{L,(p)}$ of $\mathcal{O}_{L,p}$, which is the integral closure of $\mathbb{Z}_{(p)}$ in L , is a PID. Finally, the computations for Algorithm A need not be done in L : they can be carried out in the subfields $L_i = \mathbb{Q}(\alpha_i)$ and $L_{ij} = \mathbb{Q}(\alpha_i, \alpha_j)$, working with the restriction of $\text{ord}_{\mathfrak{p}}(\cdot)$ to those fields.

Since $\alpha_1, \dots, \alpha_k$, L , and \mathfrak{p} are not known in advance, one can proceed as follows. Put $P(x) = f(x) - xg(x)$, and put $Q(x) = g(x)$ or $Q(x) = f(x) - ag(x)$ according as $a = \varphi(\infty)$ is infinite or finite. Let $f_1(x), \dots, f_r(x)$ be the distinct monic irreducible factors of $P(x)$ and $Q(x)$, so $\alpha_1, \dots, \alpha_k$ are the roots of $f_1(x), \dots, f_r(x)$. For each $j = 1, \dots, r$, put $\tilde{L}_j = \mathbb{Q}[x]/(f_j(x))$ and let $\tilde{\alpha}_j$ be the image of x in \tilde{L}_j . Find the maximal ideals $\tilde{\mathfrak{p}}_{j\ell}$ of $\mathcal{O}_{\tilde{L}_j,(p)}$ and the corresponding valuations $\text{ord}_{\tilde{\mathfrak{p}}_{j\ell}}(\cdot)$. Carry out Step (2) of Algorithm A for each pair $(\tilde{\alpha}_j, \text{ord}_{\tilde{\mathfrak{p}}_{j\ell}}(\cdot))$. Up to conjugacy, this is equivalent to carrying out Step (2) for the roots α_i of $f_j(x)$ and the valuation $\text{ord}_{\mathfrak{p}}(\cdot)$.

The minimization of $\chi_i(t)$ in Step (2e) can be done crudely in $O(d^3)$ steps by computing the intersection points of each pair of affine functions, and comparing the values of the functions at those points. It could be done more efficiently by first finding highest of the functions with given slope $m \equiv d^2 + d \pmod{2d}$, then solving for the intersection point of the functions with slopes $d^2 + d$ and $-d^2 - d$ and comparing function values at that point, and continuing on with a binary search.

If in Step (4a) the Minimal Resultant Locus turns out to be a single point (in particular if d is even) the algorithm terminates. However, if the Minimal Resultant Locus is a segment, it must either have the form $[\zeta_{D(\alpha_i, r_{i,1})}, \zeta_{D(\alpha_i, r_{i,2})}]$ for some i , or $[\zeta_{D(\alpha_i, r_{i,1})}, \zeta_{D(\alpha_i, r_{i,2})}] \cup [\zeta_{D(\alpha_j, r_{j,1})}, \zeta_{D(\alpha_j, r_{j,2})}]$ for some i and j , where $r_{i,2} = r_{j,2}$ and the segments are disjoint except for their upper endpoint. To carry out Step (4b) one should first find the segments $[\zeta_{D(\alpha_i, r_{i,1})}, \zeta_{D(\alpha_i, r_{i,2})}]$ for which $r_{i,2}$ is maximal, and among those, choose one for which $r_{i,1}$ is minimal. The corresponding segment $[\zeta_{D(\alpha_i, r_{i,1})}, \zeta_{D(\alpha_i, r_{i,2})}]$ will either be the entire Minimal Resultant Locus, or one leg of it. Suppose this segment came from the pair $(\tilde{\alpha}_1, \text{ord}_{\tilde{\mathfrak{p}}_{1,1}}(\cdot))$ and the field \tilde{L}_1 . One should then factor $f_1(x), \dots, f_r(x)$ over $\tilde{L}_1[x]$, and for each irreducible factor $f_{i,h}(x)$ (except the linear factor $x - \tilde{\alpha}_1$ of $f_1(x)$)

one should form the field $\tilde{L}_{1,i,h} = \tilde{L}_1[x]/(f_{i,h}(x))$, find the maximal ideals of $\mathcal{O}_{\tilde{L}_{1,i,h},(p)}$ lying over $\tilde{\mathfrak{p}}_{1,1}$ and carry out Step (2) again for these fields and valuations. One can then determine the relevant nodes of $\Gamma_{\text{Fix},\varphi^{-1}(a)}$, and complete Step (4b) by using them to decide whether the Minimal Resultant Locus has one leg or two.

The author has not carried out a detailed running time analysis of this procedure (which would be lengthy, and tangential to the purposes of the paper), but using the standard number-theoretic algorithms below it is evident that it could be implemented to run in probabilistic polynomial time.

Lenstra, Lenstra and Lovasz ([17]) showed that a polynomial $h(z) = a_0 + a_1z + \cdots + a_nz^n \in \mathbb{Q}[z]$ can be deterministically factored over \mathbb{Q} in $O(n^{12} + n^9 \log |h|)$ bit operations, where $|h| = (\sum_i |a_i|^2)^{1/2}$. A.K. Lenstra ([16]) proved an analogous result for polynomials over a number field. A result of Mignotte ([18], see for example Cohen [10], §3.5.1) assures that the lengths of the coefficients of the factors are polynomially bounded in terms of the input. If $F = \mathbb{Q}(\beta)$ is a number field, where β is an algebraic integer, standard methods for finding \mathcal{O}_F such as Zassenhaus's Round Two algorithm (see [10], §6.1) involve factoring the discriminant of the minimal polynomial of β , and then using linear algebra to successively enlarge the order $\mathbb{Z}[\beta]$ to be q -maximal at each prime q dividing the discriminant. There is no known polynomial time algorithm for factoring integers, but Zassenhaus's algorithm can compute a p -maximal order $\mathcal{O}_{F,p} \subset \mathcal{O}_F$ without factoring the discriminant. Since the discriminant is known, using ([10], Algorithm 2.4.6) the linear algebra computations can be done without coefficient explosion. A \mathbb{Z} -basis for $\mathcal{O}_{F,p}$ gives an integral basis for $\mathcal{O}_{F,(p)}$ over $\mathbb{Z}_{(p)}$; thus the algorithm of Buchmann-Lenstra (see [10], §6.2) can be used to find the maximal ideals \mathfrak{p} of $\mathcal{O}_{F,(p)}$ above (p) . This involves carrying out a series of matrix computations over \mathbb{F}_p . Given $0 \neq x \in \mathcal{O}_F$, the standard way to compute $\text{ord}_{\mathfrak{p}}(x)$ is to find an element $\beta \in \mathfrak{p}^{-1} \setminus \mathcal{O}_F$, and then determine the largest integer N such that $\beta^N x \in \mathcal{O}_F$ (see Cohen [10], §4.8.3). However, this can equally well be done over $\mathcal{O}_{F,(p)}$. The algorithms of Zassenhaus and Buchmann-Lenstra run in probabilistic polynomial time; the probabilistic aspect comes from the need to factor polynomials over finite fields. Using Berlekamp's algorithm ([5]) polynomials of degree ℓ in $\mathbb{F}_q[x]$ can be factored in probabilistic polynomial time $O(\ell^3 \log(q)^3)$; improvements have been given by Cantor-Zassenhaus ([9]), Kaltofen-Shoup ([15]), and others.

With suitable modifications, the procedure outlined above could be generalized to rational functions $\varphi(z)$ over arbitrary global fields, and should still run in probabilistic polynomial time. This uses that polynomials over a global field can be factored in polynomial time, as shown by Pohst and Omanã ([19], [20]) and Belabas, van Hoeij, Klüners and Steel ([6]).

Now let H_v be a local field. Suppose $\varphi(z) \in H_v(z)$ has degree $d \geq 2$, and take $K = \mathbb{C}_v$. Below, we give a “steepest descent” algorithm for finding an H_v -rational type II point where $\text{ordRes}_{\varphi}(\cdot)$ takes its least value. Working within H_v , the algorithm finds the H_v -minimum for $\text{ordRes}_{\varphi}(\cdot)$ and a $\gamma \in \text{GL}_2(H_v)$ which achieves it. The algorithm also decides whether the H_v -minimum is the absolute minimum.

The algorithm takes the path of steepest descent towards $\text{MinResLoc}(\varphi)$, starting at ζ_G . The path necessarily begins with a segment going “upward” from ζ_G towards ∞ (this segment may have length 0) to some point $\zeta_{D(0,R)} = \zeta_{D(a,R)}$, then goes “downward” from $\zeta_{D(a,R)}$ to a point $\zeta_{D(a,r)} \in \text{MinResLoc}(\varphi)$. By Proposition 3.4, at any H_v -rational

type II point outside $\text{MinResLoc}(\varphi)$, $\text{MinResLoc}(\varphi)$ lies in a direction coming from the tree spanned by $\mathbb{P}^1(H_v)$. Thus the path of steepest descent runs along the tree spanned by $\mathbb{P}^1(H_v)$ until it either reaches a point in $\text{MinResLoc}(\varphi)$, or branches off that tree between two H_v -rational type II points, one of which will minimize $\text{ordRes}_\varphi(\cdot)$ on H_v -rational type II points. The algorithm steps between H_v -rational type II points and stops when an H_v -rational type II point minimizing $\text{ordRes}_\varphi(\cdot)$ is reached.

We will assume the residue field $k_v = \mathcal{O}_v/\mathfrak{M}_v$ is isomorphic to \mathbb{F}_q , and that $\text{ord}(\cdot)$ is normalized so that $\text{ord}(\pi) = 1$ for a uniformizer π for \mathfrak{M}_v . Given $a \in \mathcal{O}_v$, we write $\bar{a} = a \pmod{\mathfrak{M}_v} \in \mathbb{F}_q$ for the residue class of a .

Algorithm B: Minimize $\text{ordRes}_\varphi(\cdot)$ on H_v -rational type II points.

Given a nonarchimedean local field H_v , and $\varphi(z) \in H_v(z)$ with $d = \deg(\varphi) \geq 2$:

- (1) [Initialize.]
 - (a) Find a normalized representation (F, G) for φ .
 - (b) Compute $R = \text{ord}(\text{Res}(F, G))$.
 - (c) Set $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
 - (d) Fix an element $\pi \in H_v$ with $\text{ord}(\pi) = 1$.
- (2) [First, go up from ζ_G towards ∞ .]
 - (a) [See if $\text{ordRes}_\varphi(\cdot)$ is locally decreasing in the direction \vec{v}_∞ .]

Write $F(X, Y) = a_d X^d + a_{d-1} X^{d-1} Y + \cdots + a_0 Y^d$,
 $G(X, Y) = b_d X^d + b_{d-1} X^{d-1} Y + \cdots + b_0 Y^d$,
 then using the criterion from Lemma 1.4,
 test whether $\begin{cases} \bar{a}_\ell = 0 & \text{for } \frac{d+1}{2} \leq \ell \leq d, \\ \bar{b}_\ell = 0 & \text{for } \frac{d-1}{2} \leq \ell \leq d. \end{cases}$
 If not, $\text{ordRes}_\varphi(\cdot)$ is not decreasing in the direction \vec{v}_∞ ; go to (3).
 If so, \vec{v}_∞ is the unique direction in which $\text{ordRes}_\varphi(\cdot)$ is decreasing;
 continue on to (3b).
 - (b) [Compute how far to go up.]

For $\ell = 0, \dots, d$, put $C_\ell = R - 2d \text{ord}(a_\ell)$, $D_\ell = R - 2d \text{ord}(b_\ell)$,
 then minimize the piecewise affine function
 $\chi(t) = \max \left(\max_{0 \leq \ell \leq d} (C_\ell + (d^2 + d - 2d\ell)t), \max_{0 \leq \ell \leq d} (D_\ell + (d^2 + d - 2d(\ell + 1)t)) \right)$.
 Let R_{new} be the minimum value of $\chi(t)$,
 and let $[M, N]$ be the subset of \mathbb{R} on which it is attained
 (necessarily $N < 0$; possibly $M = N$).
 - (c) [Test the nature of the minimum.]
 - (i) If $M = N \in \mathbb{Z}$, the new minimum is at an H_v -rational type II point:
 take a step up to that point.
 Put $\eta = \begin{bmatrix} 1 & 0 \\ 0 & \pi^{-N} \end{bmatrix}$ and find a normalized representation for (F^η, G^η) :
 Let $F_*(X, Y) = \pi^{-N} a_d X^d + \pi^{-2N} a_{d-1} X^{d-1} Y + \cdots + \pi^{-(d+1)N} a_0 Y^d$,
 $G_*(X, Y) = b_d X^d + \pi^{-N} b_{d-1} X^{d-1} Y + \cdots + \pi^{-dN} b_0 Y^d$,
 then normalize $(F_*, G_*) \rightarrow (F_{\text{new}}, G_{\text{new}})$,
 update $F \leftarrow F_{\text{new}}$, $G \leftarrow G_{\text{new}}$, $R \leftarrow R_{\text{new}}$, $\gamma \leftarrow \eta$, and go to (3).
 - (ii) If $M = N \notin \mathbb{Z}$, or if $[M, N]$ is an interval which contains no integers,

- the H_v -minimum for $\text{ordRes}_\varphi(\cdot)$ is not the absolute minimum:
 let m, n be the two integers bracketing $[M, N]$,
 put $R = \min(\chi(m), \chi(n))$ and let $k \in \{m, n\}$ be a point where
 the minimum is attained; put $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & \pi^{-k} \end{bmatrix}$, and go to (4a).
- (iii) If $[M, N]$ is an interval of positive length containing an integer k ,
 the new minimum for $\text{ordRes}_\varphi(\cdot)$ is the absolute minimum:
 put $R = R_{\text{new}}$, put $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & \pi^{-k} \end{bmatrix}$, and go to (4b).
- (3) [Iterate steps down, until the H_v -minimum is reached.]
- (a) [Limit the possible directions towards $\text{MinResLoc}(\varphi)$.]
 Put $g(z) = G(z, 1)$ and $h(z) = F(z, 1) - zG(z, 1)$,
 then find the common roots of $\bar{g}(z)$ and $\bar{h}(z)$ belonging to \mathbb{F}_q .
 If there are no such roots, the current R is minimal: go to (4b).
 If there are common roots, list them as $\{\bar{\beta}_1, \dots, \bar{\beta}_k\}$, and continue to (3b).
- (b) [Find the direction of steepest descent.]
 For each $i = 1, \dots, k$, do the following:
- (i) Let $\beta_i \in \mathcal{O}_v$ be a lift of $\bar{\beta}_i$; change coordinates by $\begin{bmatrix} 1 & \beta_i \\ 0 & 1 \end{bmatrix}$,
 putting $F_*(X, Y) = F(X + \beta_i Y, Y) - \beta_i G(X + \beta_i Y, Y)$,
 $G_*(X, Y) = G(X + \beta_i Y, Y)$.
- (ii) Write $F_*(X, Y) = a_d X^d + a_{d-1} X^{d-1} Y + \dots + a_0 Y^d$,
 $G_*(X, Y) = b_d X^d + b_{d-1} X^{d-1} Y + \dots + b_0 Y^d$,
 then using the criterion from Lemma 1.4,
 test whether $\begin{cases} \bar{a}_\ell = 0 & \text{for } 0 \leq \ell \leq \frac{d+1}{2}, \\ \bar{b}_\ell = 0 & \text{for } 0 \leq \ell \leq \frac{d-1}{2}. \end{cases}$
 If so, \vec{v}_{β_i} is the unique direction in which $\text{ordRes}_\varphi(\cdot)$ is decreasing;
 exit the loop on i , and go to (3c).
 If not, continue the loop and take the next value of i .
- If there are no more values of i , the current R is minimal; go to (4b).
- (c) [Compute how far to go down.]
 For $\ell = 0, \dots, d$, put $C_\ell = R - 2d \text{ord}(a_\ell)$, $D_\ell = R - 2d \text{ord}(b_\ell)$,
 then minimize the piecewise affine function
 $\chi(t) = \max \left(\max_{0 \leq \ell \leq d} (C_\ell + (d^2 + d - 2d\ell)t), \max_{0 \leq \ell \leq d} (D_\ell + (d^2 + d - 2d(\ell + 1)t)) \right)$.
 Let R_{new} be the minimum value of $f(t)$,
 and let $[M, N]$ be the subset of \mathbb{R} on which it is attained
 (necessarily $M > 0$; possibly $M = N$).
- (d) [Test the nature of the minimum.]
- (i) If $M = N \in \mathbb{Z}$, the new minimum is at an H_v -rational type II point:
 take a step down to that point.
 Put $\eta = \begin{bmatrix} \pi^N & \beta_i \\ 0 & 1 \end{bmatrix}$ and find a normalized representation for (F^η, G^η) :
 Let $F_{**}(X, Y) = F_*(\pi^N X, Y)$, $G_{**}(X, Y) = \pi^N G_*(\pi^N X, Y)$,
 then normalize $(F_{**}, G_{**}) \rightarrow (F_{\text{new}}, G_{\text{new}})$,
 update $F \leftarrow F_{\text{new}}$, $G \leftarrow G_{\text{new}}$, $R \leftarrow R_{\text{new}}$, $\gamma \leftarrow \gamma \cdot \eta$, and go to (3).

- (ii) If $M = N \notin \mathbb{Z}$, or if $[M, N]$ is an interval which contains no integers, the H_v -minimum for $\text{ordRes}_\varphi(\cdot)$ is not the absolute minimum: let m, n be the two integers bracketing $[M, N]$, put $R = \min(\chi(m), \chi(n))$ and let $k \in \{m, n\}$ be a point where the minimum is attained; put $\gamma = \gamma \cdot \begin{bmatrix} \pi^k & \beta_i \\ 0 & 1 \end{bmatrix}$, and go to (4a).
- (iii) If $[M, N]$ is an interval of positive length containing an integer k , the new minimum for $\text{ordRes}_\varphi(\cdot)$ is the absolute minimum: put $R = R_{\text{new}}$, put $\gamma = \gamma \cdot \begin{bmatrix} \pi^k & \beta_i \\ 0 & 1 \end{bmatrix}$, and go to (4b).
- (4) [Output whether the H_v -minimum for $\text{ordRes}_\varphi(\cdot)$ is the absolute minimum.]
 - (a) Output “The H_v -minimum for $\text{ordRes}_\varphi(\cdot)$ is not the absolute minimum”, and go to (5).
 - (b) Output “The H_v -minimum for $\text{ordRes}_\varphi(\cdot)$ is the absolute minimum”, and continue on to (5).
- (5) [Output R and γ , and halt.]
 - Output “The minimal value of $\text{ordRes}_\varphi(\cdot)$ on H_v -rational type II points = R ”;
 - Output “ $\gamma = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a matrix for which $\text{ordRes}(\varphi^\gamma) = R$ ”. Halt.

The correctness of the algorithm and the fact that it terminates follow from Theorem 0.1, Lemma 1.4 and Proposition 3.4, but perhaps some remarks are in order.

After each step to a new H_v -rational type II point, the algorithm changes coordinates to bring that point back to ζ_G . The corresponding coordinate change matrices are affine, so they preserve the direction \vec{v}_∞ . This means that in Lemma 1.4 we can use the tangent directions $\vec{v}_\infty, \vec{v}_\beta$ at ζ_G , rather than the tangent directions $\vec{v}_{Q, \eta(\infty)}, \vec{v}_{Q, \eta(\beta)}$ at $Q = \eta(\zeta_G)$.

If the path of steepest descent branches off the tree spanned by $\mathbb{P}^1(H_v)$, when the algorithm moves between the two H_v -rational type II points adjacent to $\text{MinResLoc}(\varphi)$, $\text{ordRes}_\varphi(\cdot)$ will initially decrease, then increase. The stopping criteria in Steps (2c), (3a), (3b) and (3d) assure that a point where the minimum is taken is chosen.

In Step (3a), it cannot be that both $\bar{g}(z) \equiv 0$ and $\bar{h}(z) \equiv 0$, as in that case $\bar{f}(z) = \bar{h}(z) + z\bar{g}(z) \equiv 0$. Since the coefficients of f and g are the same as those of F and G respectively, this would mean (F, G) was not normalized, contrary to its construction. To motivate Step (3a), note that in Step (3b), $g(\beta_i)$ is the coefficient of Y^d in $G_*(X, Y)$ and $h(\beta_i)$ is the coefficient of Y^d in $F_*(X, Y)$. If $\overline{g(\beta_i)} \neq 0$, then in Step (3b) the coefficient \bar{b}_0 would be nonzero, and the test in Step (3b) would fail. Likewise, if $\overline{h(\beta_i)} \neq 0$, then $\bar{a}_0 \neq 0$, and again the test would fail.

In Step (3c), the matrix $\eta = \begin{bmatrix} \pi^N & \beta_i \\ 0 & 1 \end{bmatrix}$ makes the step from ζ_G to the H_v -rational type II point $\zeta_{D(\beta_i, |\pi|^N)}$. This coordinate change is realized as the composite of two partial steps, using $\eta = \begin{bmatrix} 1 & \beta_i \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \pi^N & 0 \\ 0 & 1 \end{bmatrix}$.

Algorithm B is content with finding one point where the H_v -minimum is attained. By incorporating additional tests and an extra search based on the criteria in Lemma 1.4(B), it could easily be modified to find all H_v -rational type II points where the H_v -minimum was attained. We leave this modification to the reader.

In implementing Algorithm B it is not necessary to work in a local field. If $\varphi(z)$ is defined over a number field H , and $\text{ord}_v(\cdot)$ is a nonarchimedean valuation of H (specified, for example, by giving a p -maximal order $\mathcal{O}_{H,p} \subset \mathcal{O}_H$ and a prime ideal \mathfrak{p}_v of $\mathcal{O}_{H,p}$ above p), one could carry out the algorithm using computations in H using ideas similar to those discussed in Algorithm A.

We will now discuss its running time when $H = \mathbb{Q}$ and $v = p$ is a rational prime. For simplicity, assume that $\varphi(x) = f_0(x)/g_0(x)$ is the quotient of relatively prime polynomials $f_0(x), g_0(x) \in \mathbb{Z}[x]$, where the coefficients of f_0 and g_0 have absolute value at most B . Let (F_0, G_0) be the initial normalized representation of φ from Step (1a), and let $R_0 = \text{ord}_p(\text{Res}(F_0, G_0))$ be the ord-value of its resultant, computed in Step (1b). The Hadamard bound for the archimedean size of $\text{Res}(F_0, G_0)$ is $(d+1)^d B^{2d}$, so

$$R_0 \leq d \log_p(d+1) + 2d \log_p(B) .$$

Each time Step 2 or Step 3 is executed, the distance from ζ_G to the \mathbb{Q}_p -rational type II point being considered increases by at least 1, so by Theorem 0.1, the algorithm terminates after at most $\frac{2}{d-1} R_0$ passes through Steps 2 and 3. At all intermediate stages, the coefficients of F and G remain in \mathbb{Z} ; by Theorem 0.2, it suffices to compute them modulo p^{4R_0} , and as the algorithm proceeds, the required precision decreases. Step (3a) limits the number of residue classes considered in Step (3b) to at most $d+1$; using Berlekamp's algorithm Step (3a) can be carried out in $\mathcal{O}(d^3 \log(p)^3)$ bit operations. From these considerations one sees that Algorithm B runs in probabilistic polynomial time.

5. THE CASE $d = 1$

For completeness, in this section we consider $\text{ordRes}_\varphi(\cdot)$ when $d = 1$, that is, when $\varphi(z) = \frac{f_1 z + f_0}{g_1 z + g_0} \in K(z)$ with $f_1 g_0 - f_0 g_1 \neq 0$. It is no longer true that $\text{MinResLoc}(\varphi)$ is a point or a segment of finite path-length: the reason for the difference is the simple fact that $1^2 - 1 = 0$, whereas $d^2 - d > 0$ when $d \geq 2$.

As is well known, there are three cases to consider:

- (1) $\varphi(z) = z$;
- (2) $\varphi(z)$ has (exactly) two distinct fixed points, in which case there are a $\gamma \in \text{GL}_2(K)$ and a $C \in K^\times$ with $|C| \leq 1$ and $C \neq 1$ such that $\varphi^\gamma(z) = Cz$;
- (3) $\varphi(z)$ has a single fixed point of multiplicity 2, in which case there are a $\gamma \in \text{GL}_2(K)$ and a $0 \neq C \in K$ such that $\varphi^\gamma(z) = z + C$.

By standard computations in linear algebra, it is easy to distinguish between the cases, and to find a γ which carries out the desired conjugacy: the second case occurs when the Jordan form of the matrix corresponding to φ is $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ with $\lambda \neq \mu$, and the eigenvalues are ordered so that $|\lambda| \leq |\mu|$; the third case when it is $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. In the second case $C = \lambda/\mu$, in the third case $C = 1/\lambda$. If φ and the eigenvalues are rational over a subfield $H \subset K$, then γ can be chosen to belong to $\text{GL}_2(H)$.

We will need some terminology. Given points $x_0 \neq x_1 \in \mathbb{P}^1(K)$, the *strong tube of radius R around the path $[x_0, x_1]$* is the set

$$T_{[x_0, x_1]}(R) = [x_0, x_1] \cup \{z \in \mathbb{H}_{\text{Berk}} : \rho(z, x) \leq R \text{ for some } x \in [x_0, x_1]\} .$$

If $z \in \mathbb{P}_{\text{Berk}}^1$ corresponds to a sequence of nested discs $\{D(a_i, r_i)\}_{i \geq 1}$ by Berkovich's classification theorem (see [2], p.5), we define $\text{diam}_\infty(z) = \lim_{i \rightarrow \infty} r_i$; we put $\text{diam}_\infty(\infty) = \infty$. The *horodisc of codiameter R , tangent to the point ∞* , is the set

$$H_\infty(R) = \{ z \in \mathbb{P}_{\text{Berk}}^1 : \text{diam}_\infty(z) \geq R \}.$$

The only type I point belonging to $H_\infty(R)$ is ∞ ; a point $\zeta_{D(a,r)}$ of type II or III belongs to $H_\infty(R)$ if and only if $r \geq R$. For each $a \in K$, the intersection of the path $[a, \infty]$ with $H_\infty(R)$ is the ray $[\zeta_{D(a,R)}, \infty]$. For each $S > R$, the point $\zeta_{D(0,S)}$ belongs to $H_\infty(R)$; if $a \in K$ and $|a| \leq S$, the intersection of $[a, \zeta_{D(0,S)}]$ with $H_\infty(R)$ is

$$\{ z \in [a, \zeta_{D(0,S)}] : \rho(\zeta_{D(0,S)}, z) \leq \log(S/R) \}.$$

Thus $H_\infty(R)$ can be described informally as “the set of points in $\mathbb{P}_{\text{Berk}}^1$ accessible by moving the ray $[\zeta_{D(0,R)}, \infty]$ without stretching, keeping it anchored at ∞ ”. For an arbitrary $x_0 \in \mathbb{P}^1(K)$, a *horodisc tangent to x_0* is a set of the form $\gamma(H_\infty(R))$ for some R , where $\gamma \in \text{GL}_2(K)$ is such that $\gamma(\infty) = x_0$.

Theorem 5.1. *Suppose $\varphi(z) \in K(z)$ has degree $d = 1$. The function $\text{ordRes}_\varphi(\cdot)$ on type II points extends to a function $\text{ordRes}_\varphi : \mathbb{P}_{\text{Berk}}^1 \rightarrow [0, \infty]$ which is piecewise affine and convex upwards on each path in $\mathbb{P}_{\text{Berk}}^1$, with respect to the logarithmic path distance. It is finite and continuous on \mathbb{H}_{Berk} with respect to the strong topology, and achieves its minimum on a nonempty set $\text{MinResLoc}(\varphi) \subset \mathbb{P}_{\text{Berk}}^1$. Furthermore*

- (1) *If $\varphi(z) = z$, then $\text{ordRes}_\varphi(\cdot) \equiv 0$ and $\text{MinResLoc}(\varphi) = \mathbb{P}_{\text{Berk}}^1$.*
- (2) *If $\varphi(z)$ has exactly two fixed points x_0, x_1 , let $\gamma \in \text{GL}_2(K)$ and $C \in K^\times$ with $|C| \leq 1$, $C \neq 1$, be such that $\varphi^\gamma(z) = Cz$. The minimal value of $\text{ordRes}_\varphi(\cdot)$ is $\text{ord}(C)$, and φ has potential good reduction if and only if $|C| = 1$. When $|C| < 1$, or when $|C| = 1$ and $|C - 1| = 1$, then $\text{MinResLoc}(\varphi)$ is the path $[x_0, x_1]$. When $|C| = 1$ and $|C - 1| < 1$, put $R = \text{ord}(C - 1) > 0$; then $\text{MinResLoc}(\varphi)$ is the strong tube $T_{[x_0, x_1]}(R)$. The function $\text{ordRes}_\varphi(\cdot)$ takes the value ∞ at each point of $\mathbb{P}^1(K) \setminus \{x_0, x_1\}$, and is continuous on $\mathbb{P}_{\text{Berk}}^1 \setminus \{x_0, x_1\}$ relative to the strong topology.*
- (3) *If $\varphi(z)$ has one fixed point x_0 , let $\gamma \in \text{GL}_2(K)$ and $0 \neq C \in K$ be such that $\varphi^\gamma(z) = z + C$. Then the minimal value of $\text{ordRes}_\varphi(\cdot)$ is 0 and φ has potential good reduction. Put $R = |C|$. Then $\text{MinResLoc}(\varphi)$ is the horodisc tangent to x_0 given by $\gamma(H_\infty(R))$. The function $\text{ordRes}_\varphi(\cdot)$ takes the value ∞ at each point of $\mathbb{P}^1(K) \setminus \{x_0\}$, and is continuous on $\mathbb{P}_{\text{Berk}}^1 \setminus \{x_0\}$ relative to the strong topology.*

Proof. The fact that $\text{ordRes}_\varphi(\cdot)$ extends from type II points to a function $\text{ordRes}_\varphi : \mathbb{P}_{\text{Berk}}^1 \rightarrow [0, \infty]$ which is piecewise affine and convex upwards on each path in $\mathbb{P}_{\text{Berk}}^1$ with respect to the logarithmic path distance, and is finite and continuous on \mathbb{H}_{Berk} with respect to the strong topology, follows by the same argument as in the proof Theorem 0.1. Indeed, $\text{ordRes}_\varphi(\cdot)$ is Lipschitz continuous on \mathbb{H}_{Berk} , with Lipschitz constant $1^2 + 1 = 2$. To prove the remaining assertions, we will make explicit computations in each case.

When $\varphi(z) = z$, it is easy to see that $\varphi^\gamma(z) = z$ for each $\gamma \in \text{GL}_2(K)$, and the assertions in part (1) of the Theorem follow trivially.

Next assume φ has exactly two distinct fixed points x_0, x_1 , and let $\gamma \in \text{GL}_2(K)$ be such that $\varphi^\gamma(z) = Cz$ with $|C| \leq 1$, $C \neq 1$. After relabeling x_0, x_1 if necessary, we can assume that $\gamma(0) = x_0$ and $\gamma(\infty) = x_1$. Given $A \in K^\times$ and $B \in K$, put $\tau = \tau_{A,B} = \begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix}$.

As A and B vary, the points $\zeta_{D(B,|A|)} = \tau_{A,B}(\zeta_G)$ range over all type II points in \mathbb{H}_{Berk} . Consider the representation $(F^\gamma(X, Y), G^\gamma(X, Y)) = (CX, Y)$ for φ^γ . One sees easily that $\text{ordRes}(\varphi^\gamma) = \text{ord}(C)$ and $(F^{\gamma\tau}, G^{\gamma\tau}) = (ACX + B(C-1)Y, AY)$, which gives

$$(26) \quad \text{ordRes}_{\varphi^\gamma}(\zeta_{D(B,|A|)}) = \max(\text{ord}(C), \text{ord}(C) - 2\text{ord}(B) - 2\text{ord}(C-1) + 2\text{ord}(A)) .$$

When $|C| < 1$, or when $|C| = |C-1| = 1$, formula (26) simplifies to

$$\text{ordRes}_{\varphi^\gamma}(\zeta_{D(B,|A|)}) = \max(\text{ord}(C), \text{ord}(C) + 2\text{ord}(A) - 2\text{ord}(B)) .$$

When $B = 0$, then $\text{ordRes}_{\varphi^\gamma}(\zeta_{D(0,|A|)}) = \text{ord}(C)$ for all A , so $\text{ordRes}_{\varphi^\gamma}(\cdot) \equiv \text{ord}(C)$ on the path $[0, \infty]$. Next suppose $B \neq 0$. The path $[B, \infty]$ meets $[0, \infty]$ at $\zeta_{D(0,|B|)}$, and for $|A| \leq |B|$ we see that $\text{ordRes}_{\varphi^\gamma}(\zeta_{D(B,|A|)}) = \text{ord}(C) - 2\text{ord}(A/B) > \text{ord}(C)$. Thus $\text{ordRes}_{\varphi^\gamma}(\cdot)$ increases as one moves away from $[0, \infty]$, and $\text{ordRes}_{\varphi^\gamma}(B) = \infty$. It follows that $\text{MinResLoc}(\varphi^\gamma) = [0, \infty]$ and that $\text{ordRes}_{\varphi^\gamma}(x) = \infty$ for all $x \in \mathbb{P}^1(K) \setminus \{0, \infty\}$. By Proposition 1.3, $\text{ordRes}_{\varphi^\gamma}(\cdot)$ is continuous on $\mathbb{P}_{\text{Berk}}^1 \setminus \{0, \infty\}$ relative to the strong topology.

When $|C-1| < 1$, formula (26) becomes

$$\text{ordRes}_{\varphi^\gamma}(\zeta_{D(B,|A|)}) = \max(0, -2\text{ord}(C-1) + 2\text{ord}(A) - 2\text{ord}(B)) .$$

When $B = 0$, then $\text{ordRes}_{\varphi^\gamma}(\zeta_{D(0,|A|)}) = 0$ for all A , so $\text{ordRes}_{\varphi^\gamma}(\cdot) = 0$ on $[0, \infty]$. When $B \neq 0$, for $|A| \leq |B|$ we see that $\text{ordRes}_{\varphi^\gamma}(\zeta_{D(B,|A|)}) = 0$ if $\text{ord}(A/B) \leq \text{ord}(C-1)$, while $\text{ordRes}_{\varphi^\gamma}(\zeta_{D(B,|A|)}) = -2\text{ord}(C-1) + 2\text{ord}(A/B) > 0$ if $\text{ord}(A/B) > \text{ord}(C-1)$. Putting $R = \text{ord}(C-1)$ we see that $\text{MinResLoc}(\varphi^\gamma)$ is the strong tube $T_{[0, \infty]}(R)$ and that $\text{ordRes}_{\varphi^\gamma}(x) = \infty$ for all $x \in \mathbb{P}^1(K) \setminus \{0, \infty\}$. By Proposition 1.3, $\text{ordRes}_{\varphi^\gamma}(\cdot)$ is continuous on $\mathbb{P}_{\text{Berk}}^1 \setminus \{0, \infty\}$ relative to the strong topology. Transferring these assertions back to φ using formula (22), we obtain part (2) of the Theorem.

Finally suppose φ has exactly one fixed point x_0 . Let $\gamma \in \text{GL}_2(K)$ be such that $\varphi^\gamma(z) = z + C$ with $C \neq 0$; then $\gamma(\infty) = x_0$. Given $A \in K^\times$ and $B \in K$, let $\tau = \tau_{A,B}$ be as above. Consider the representation $(F^\gamma(X, Y), G^\gamma(X, Y)) = (X + CY, Y)$ for φ^γ . Then $\text{ordRes}(\varphi^\gamma) = 0$ and $(F^{\gamma\tau}, G^{\gamma\tau}) = (AX + CY, AY)$, which gives

$$(27) \quad \text{ordRes}_{\varphi^\gamma}(\zeta_{D(B,|A|)}) = \max(0, 2\text{ord}(A/C)) .$$

Put $R = |C|$. For each $B \in K$, formula (27) shows that $\text{ordRes}_{\varphi^\gamma}(\zeta_{D(B,|A|)}) = 0$ if $|A| \geq |C|$, while $\text{ordRes}_{\varphi^\gamma}(\zeta_{D(B,|A|)}) = 2\text{ord}(A/C) > 0$ if $|A| < |C|$. Thus $\text{MinResLoc}(\varphi^\gamma)$ is the horodisc $H_\infty(R)$, and $\text{ordRes}_{\varphi^\gamma}(x) = \infty$ for all $x \in \mathbb{P}^1(K) \setminus \{\infty\}$. By Proposition 1.3, $\text{ordRes}_{\varphi^\gamma}(\cdot)$ is continuous on $\mathbb{P}_{\text{Berk}}^1 \setminus \{\infty\}$ relative to the strong topology. Transferring these assertions back to φ using formula (22), we obtain part (3) of the Theorem. \square

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ROBERT RUMELY, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GEORGIA 30602, USA

E-mail address: rr@math.uga.edu